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WIDER Working Paper 2019/55

Counting-based multidimensional poverty identification

From deprivation weights to bundles

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August 2019

United Nations University World Institute for Development Economics Research

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Abstract: In the widely used class of multidimensional poverty measures introduced by Alkire and Foster (2011), dimension-specific weights combined with a single cut-off parameter play a fundamental role in identifying who is multidimensionally poor. This paper revisits how these parameters are understood, revealing they do not uniquely characterise who is identified as poor and that the weights do not reliably reflect each dimensions' relative importance. Drawing on insights from Boolean algebra, I demonstrate that the set of 'minimum deprivation bundles' constitutes an intuitive and unique characterization of Alkire-Foster identification functions. This provides a formal foundation for various analytical innovations, namely: a novel poverty decomposition based only on the unique properties of each identification function; and metrics of dimensional power, which capture the effective importance or 'value' of each dimension across all possible combinations of deprivations. These insights are illustrated using deprivation data from Mozambique and applying various identification functions, including a close replica of the international MPI (multidimensional poverty index).

Key words: Boolean functions, composite indexes, Mozambique, multidimensional

poverty, weighting

JEL classification: I32, O55, C15

Acknowledgements: Thanks to Paul Anand, Paola Ballon, James Foster, Simon Quinn, Vincenzo Salvucci, Ricardo Santos, Suman Seth, Finn Tarp, Erik Thorbecke and Gaston Yalonetzky for comments and encouragement on my investigation of this topic. All errors and omissions are my own.

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This study has been prepared within the UNU-WIDER project 'Inclusive growth in Mozambique – scaling-up research and capacity'.

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ISSN 1798-7237 ISBN 978-92-9256-689-0

https://doi.org/10.35188/UNU-WIDER/2019/689-0

Typescript prepared by the author.

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The Institute is funded through income from an endowment fund with additional contributions to its work programme from Finland, Sweden, and the United Kingdom as well as earmarked contributions for specific projects from a variety of donors.

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1 Introduction

The conceptualization and measurement of well-being as a multidimensional phenomenon is long-standing. Following the seminal contribution of Alkire and Foster (2011a), who proposed a generalization of the Foster-Greer-Thorbecke class of poverty measures (Foster et al., 1984) to multiple domains of deprivation, their approach to measuring multidimensional well-being has been widely applied. A Google Scholar search reveals over 1,000 papers applying or discussing their framework.[1] Also, since 2010, the UN's annual Human Development Report has included a multidimensional poverty index of the Alkire-Foster type, based on survey data for more than 100 countries and covering ten dimensions of well-being. And both Mexico and Colombia have adopted the same type of multidimensional index to guide and track progress in national poverty reduction (e.g., Angulo et al., 2016).

In line with many approaches to multidimensional poverty measurement, as well as the construction of various composite indexes (e.g., Foster et al., 2013; Jones, 2016), the Alkire-Foster procedure aggregates observations across various separate dimensions by applying a set of dimensional weights (Decancq and Lugo, 2013).[2] In addition to these, an overall cut-off (threshold) is applied to determine whether a given weighted sum of deprivations is sufficient to identify a unit as poor. Reflecting the central role of these weights and cut-off in the procedure, it is natural that different identification schemes (functions) are almost invariably described with recourse to the choice of these inputs. This is exemplified by debates over how identification functions should be constructed, which focus on choosing specific numerical values for the weights and cut-off (e.g., Alkire and Foster, 2011b; Alkire et al., 2011; Ravallion, 2012; Santos and Villatoro, 2018; Abdu and Delamonica, 2018; Mitra, 2018). In doing so, the assumption is that variations in the weights and/or cut-off map in transparent fashion to differences in who is identified as poor. Consequently, these same inputs are often thought to be appropriate objects for participatory determination and public debate (e.g., Sen and Anand, 1997).

The above indicates that the weights and cut-offs are generally treated as sufficient to characterise poverty identification functions within the Alkire-Foster tradition. In addition, the same weights are frequently interpreted as capturing the relative importance of each dimension. As Alkire and Santos (2014) state: "the weighting structure determines the assumed trade-offs across deprivations" (p. 256; also see Alkire and Foster, 2011a). Similarly, in an application of the Alkire-Foster approach, Pasha (2017) notes that: "[w]eights for any composite index of well-being can be based on the trade-offs they imply between the dimensions of well-being." (p. 270). With a similar motivation in mind, Decancq and Lugo (2013) provide a detailed survey of different approaches to setting weights and their underlying rationales, also concluding that weights are crucial factors that determine the trade-off between dimensions.

[1] These results are based on the search string: ' "multidimensional poverty" "Alkire Foster" ' (16 April 2019).

[2] Key contributions to the literature on constructing multidimensional measures of well-being include Maasoumi (1986); Atkinson (2003); Bourguignon and Chakravarty (2003). For a broader discussion see Anand et al. (2011).

This paper does not contest the central role that the vector of weights and cut-off play within multi-dimensional poverty identification in the Alkire-Foster tradition. Nonetheless, the purpose here is to clarify how these inputs should be interpreted and deployed in auxiliary analyses. Contrary to their general treatment in the literature, I show that different weight/cut-off vectors are not unique to each poverty identification function. Theoretically, an infinite number of weights/cut-offs will yield identical identification functions and, thereby, count the same units as poor. This is important since it means variations in the weights or cut-off do not map directly to differences in identification outcomes; it also means that the numerical values of weights do not provide clear guidance as to the relative importance of each deprivation dimension. Thus, an exclusive focus on the weights/cut-off in both the design and analysis of different identification functions may be at best inefficient, and at worst misleading.

To develop these arguments, Section 2 begins with a motivating example. It shows, numerically and graphically, how superficially different weight/cut-off vectors map to the same identification outcomes. This holds not only for the aggregate poverty headcount, but also at the unit individual level – i.e., all the same units are identified as either poor or non-poor by multiple different input vectors. Moreover, I further demonstrate this result does not depend on minuscule changes to the magnitudes of the weights/cut-offs; rather, dissimilar rankings of the dimensional weights can yield equivalent identification outcomes.

Section 3 explains how this result comes about. To do so, I highlight the Alkire-Foster procedure constitutes an application of Boolean threshold logic, the functional properties of which have been extensively studied and arise in other contexts, including weighted voting games. Drawing on this literature, the number of feasible poverty identification functions in m variables is established as finite. In addition, I demonstrate a simple means to uniquely characterise each identification function, based on what I refer to as the set of 'minimal deprivation bundles'. This is the collection of the smallest bundles of deprivations required to identify any unit as poor, and which is the direct counterpart of minimal winning coalitions in voting games.

Section 4 pursues two (of various) applications that derive from the re-presentation of Alkire-Foster identification functions in terms of minimal bundles of deprivations. The first is a novel poverty decomposition, which dispenses with the non-unique weights/cut-off to reveal both the absolute contribution of each minimal bundle to the aggregate headcount as well as the contribution of each dimension. Using data from Mozambique, I show that this unique decomposition can differ sharply to the conventional decompositions found in applications of the Alkire-Foster procedure. The second application presents metrics of dimensional power, based on a number of indexes used extensively in game theory. This yields new insights about the relative importance of each dimension, which diverges from the magnitudes of the weights. Furthermore, I note that given the formal properties of these power metrics they can be broadly interpreted as shadow values. Lastly, Section 5 concludes and notes future directions of research.

2 Motivating example

This section outlines some of the difficulties that arise when identification of the poor is considered purely in terms of selecting a set of weights and an associated cut-off. Before proceeding, some basic definitions are in order. As per the conventional Alkire-Foster set-up, assume there exists an $n \times m$ matrix of binary deprivation measures, with n units of interest (e.g., individuals) and m dimensions of deprivation. The elements of this matrix are labelled d_{ij} such that $d_{ij} = 1$ if unit $i \in \{1, \ldots, n\}$ is deemed to be deprived in dimension $j \in \{1, \ldots, m\}$ and zero otherwise.[3] To identify who is poor, a vector of normalized weights $(w_1, w_2, ..., w_m)$ and single cut-off, k, is conventionally used. Concretely, unit i's binary poverty status is calculated by applying the indicator function:

$$h_i = \mathbf{1}\left[\sum_{j=1}^{m} d_{ij} w_j \geq k\right] \tag{1}$$

$$\text{where } \forall j : 0 < w_j < 1, \ \sum_{j=1}^{m} w_j = 1, \ 0 < k \leq 1$$

Equation (1) identifies if unit i is poor according to a given vector of weights and cut-off. In this sense, a particular combination $(\vec{w}; k)$ constitutes a poverty identification scheme (poverty definition). In turn, the aggregate multidimensional poverty headcount is obtained from the (sample weighted) average of the resultant vector of unit-specific poverty indicators: $H = \mathrm{E}(h_i)$; and the adjusted headcount is derived as: $M_0 = \mathrm{E}(h_i \times \sum_{j=1}^{m} d_{ij} w_j)$, which takes into account differences in the intensity of poverty among the poor.

The above highlights the key roles of the weight vector and cutoff for poverty identification. An overlooked feature of the Alkire-Foster approach is that what may appear quite distinct choices for $(\vec{w}; k)$ can map to the exact *same* poverty identification outcomes for the same input vector – i.e., different choices of $(\vec{w}; k)$ typically do not equate to unique poverty definitions. This feature is illustrated in Table 1 below, which simulates a case of five deprivation dimensions and where each row describes a particular choice for $(\vec{w}; k)$. The first five columns indicate the weights ascribed to each dimension, and the sixth column gives the cut-off. Row 1(a) is a naïve scheme that ascribes equal weights to each dimension; so, with $k = 0.8$, a unit (individual) must be deprived in at least four dimensions to be considered poor. Rows 1(b) and 1(c) present alternative weights and cut-offs (generated using pseudo-random numbers); but both of these functions map to exactly the same poverty definition as 1(a) – i.e., the sum of any four of the weights (and no less) is always required to breach the associated cut-off.

[3] I presume throughout that both the deprivation dimensions and unit-wise observations are held constant – i.e., are taken from the raw data. Note the binary deprivation measures can be obtained in a variety of ways, including via transformations of continuous measures.

Table 1: Examples of equivalent poverty identification functions

f	w_1	w_2	w_3	w_4	w_5	k	H	M_0
1(a)	0.200	0.200	0.200	0.200	0.200	0.800	0.346	0.307
1(b)	0.179	0.156	0.222	0.235	0.207	0.684	0.346	0.303
1(c)	0.165	0.221	0.184	0.211	0.220	0.673	0.346	0.307
c.v.	0.097	0.171	0.096	0.084	0.048	0.098	0.000	0.007
2(a)	0.328	0.047	0.459	0.068	0.099	0.515	0.490	0.422
2(b)	0.300	0.027	0.483	0.050	0.140	0.514	0.490	0.427
2(c)	0.136	0.114	0.409	0.148	0.194	0.536	0.490	0.398
c.v.	0.408	0.724	0.083	0.588	0.331	0.024	0.000	0.036

Note: Each row sets out the parameters of an Alkire-Foster poverty identification function in five dimensions, containing a set of weights $(w_1, \ldots, w_5)$ and cut-off (k); columns H and M_0 report the aggregate headcount and adjusted headcount, respectively, based on the same parameters and using data from Mozambique in 2014/15 (see Section 4).

Source: author's calculations.

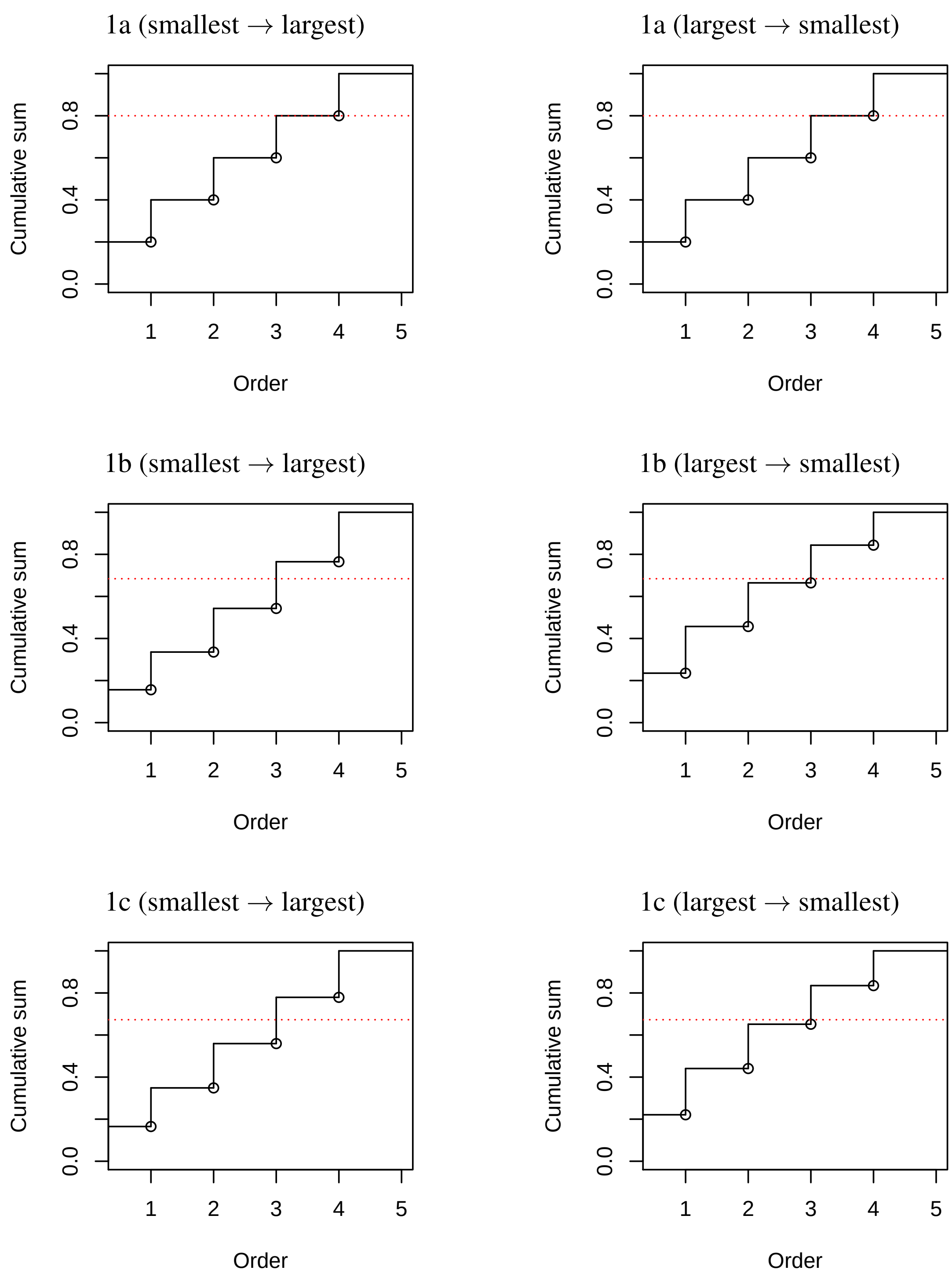
The mechanics behind this result, which prefigure results to come, are demonstrated in Table A1 in the Supplementary Material. Here I enumerate all $2^5 = 32$ unique combinations of deprivations in five dimensions, which represents the basis for a Boolean truth table. For each combination, I apply equation (1) using each $(\vec{w}; k)$ from rows 1(a)-1(c) and verify whether the given combination of deprivations is 'sufficient' to identify a unit as poor (i.e., if the sum of weights associated with that combination exceeds the cut-off). This exercise confirms that rows 1(a)-1(c) classify all possible combinations in exactly the same way. So, it is no surprise that the overall poverty headcounts (H) reported in the table, based on survey data from Mozambique (see Section 4 for details) are identical.[4]

The same idea is illustrated visually in Figure 1, which plots the cumulative sum of the weights, ordered smallest to largest and *vice versa* for each identification function. In all cases, four steps (weights) are required either to exceed or to equal the cut-off, indicated by the dashed horizontal line. It is precisely because poverty identification can be described via a (non-smooth) step function that there is a scope for alternative values of $(\vec{w}; k)$ to yield equivalent identification conditions. And here there is non-trivial variation in the weights ascribed to each dimension (and the cut-off), suggested by the coefficient of variation (c.v.) for each column.

Rows 2(a)-(c) provide a second example of different choices of $(\vec{w}; k)$ that map to equivalent poverty

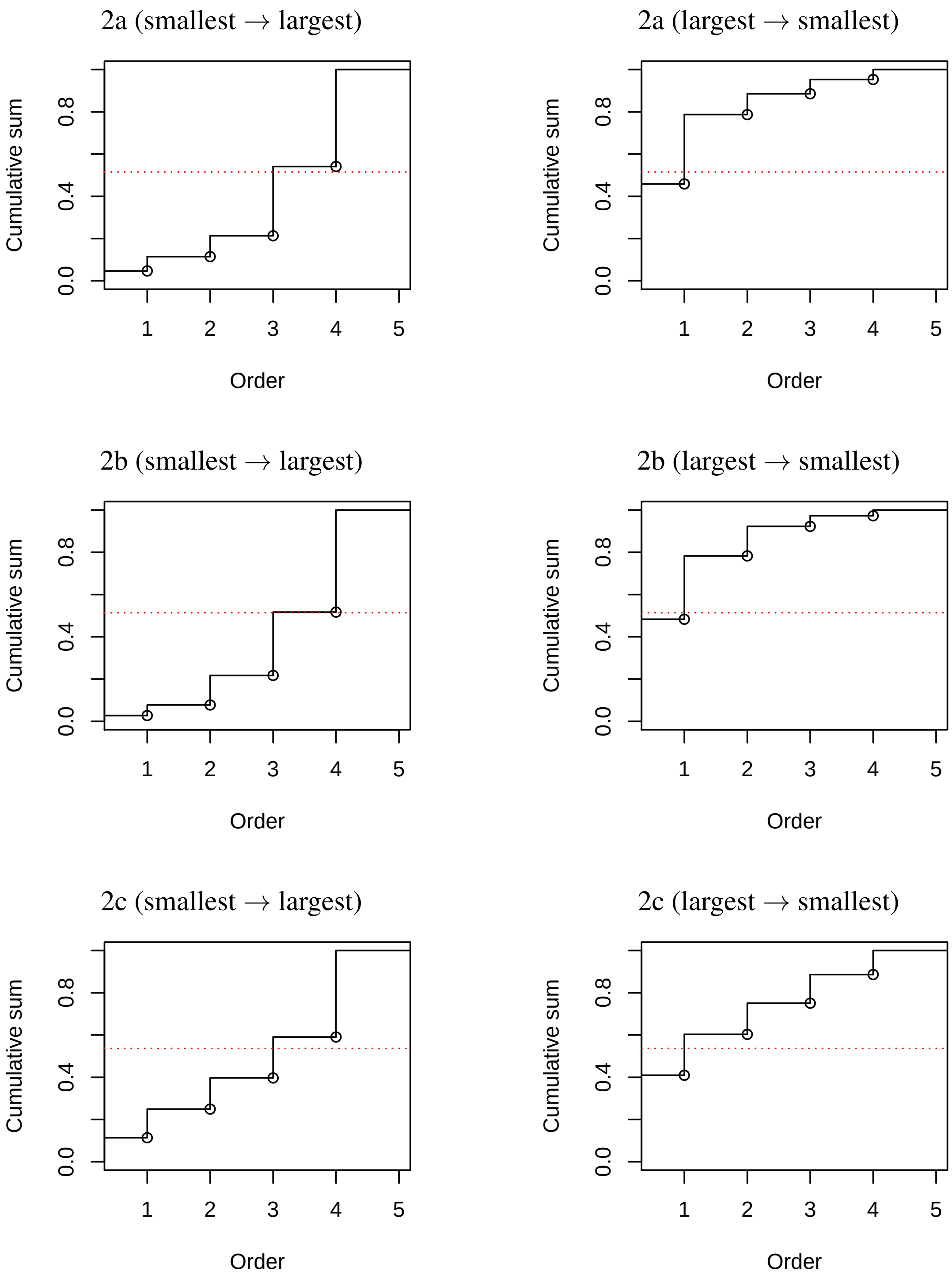
[4] Note that differences in weights have a direct influence on the M_0 metric, although even here the variation appears small. The issue is that while multiple weights and cut-offs map to the *same* identification function, it remains the case that within the set of equivalent weights and cut-offs (identifying the same people as poor), the measure M_0 will vary as long as the chosen weights (as opposed to deprivation shares) are deployed in the calculation.

Figure 1: Visualization of poverty identification functions



Note: Each figure plots the cumulative sum of weights based on the vectors reported in Table 1 (rows 1a-1c), ordered from smallest to largest and *vice versa*; the associated cut-off k is given by the dashed horizontal line. Source: author's illustration; see text.

Figure 2: Visualization of poverty identification functions



Note: Each figure plots the cumulative sum of weights based on the vectors reported in Table 1 (rows 2a-2c), order from smallest to largest and *vice versa*; the associated cut-off k is given by the dashed horizontal line.
Source: author's illustration; see text.

definitions, visualised in Figure 2. The combinations of weights that are sufficient to exceed the cut-off are more varied here, ranging from four to just two. Also in keeping with the first example, the rank order of the weights in each dimension is not consistent between rows; and the relative magnitude of the weights varies substantially. For instance, in Row 2(a) the ratio w_1/w_5 is greater than three; but in Row 2(c), the same ratio is less than one. Despite these large differences, there remains no variation in the contribution of each dimension toward poverty identification and each row vector $(\vec{w}; k)$ identifies the same units as poor or non-poor (see also Table A1).

These examples reveal that effective differences in poverty identification do not map directly from differences in the values of $(\vec{w}; k)$. As such, the impact of variations in the weights ot cut-off on the identification function are not immediately obvious; and the relative importance of each dimension to identifying who is poor does not appear to be captured by distances between weights or their rank order. From this, it follows that an exclusive focus on the numerical magnitudes of weights and cut-offs may give a false sense of precision and, in some instance, may even be misleading.

3 Identification via deprivation bundles

The previous section demonstrated that a given combination of weights and cut-off $(\vec{w}; k)$ does not necessarily map to a unique definition of *who* is poor. This section clarifies why this is the case and, in turn, shows how poverty identification functions can be uniquely characterised. The starting point is the previously un(der)recognised isomorphism between the Alkire-Foster poverty identification function and Boolean threshold functions. The isomorphism is revealed by the obvious equivalence between equation (1) and equation (10) in the Appendix, which defines Boolean threshold functions and describes their key features. A primary theoretical result concerning positive Boolean threshold functions is that each can be uniquely represented by the disjunction of its prime implicants, also known as its complete disjunctive normal form (DNF) or Blake canonical form. This also is shown formally in the Appendix.[5] Prime implicants of a Boolean threshold function are collections of elements (inputs taking the value of zero or one), all of which must be true (equal to one) for the aggregate binary outcome to also be true and none of which are redundant, meaning that if any one element were false (zero) then the aggregate function could not be true. For example, taking the case of equal weights described in row 1(a) of Table 1, a prime implicant would be a collection of any four of the dimensions of deprivation. They are 'prime' because at least four dimensions is always needed for a unit to be considered poor; and the collection of all five dimensions is non-prime because the unit need not be deprived in all five to be considered poor.

[5] The Appendix draws on standard textbook material, principally Crama and Hammer (2011) (*viz.*, Theorems 1.04, 1.13 and 1.23) which provides a comprehensive general treatment. The definitions contained therein are valuable since Boolean functions are not widely studied by social scientists, especially outside of game theory and formal logic.

The equivalence between Boolean threshold functions and the Alkire-Foster procedure implies the latter is just one of various applications of the former, which also include weighted voting games (e.g., Taylor and Zwicker, 1992; de Keijzer et al., 2012). So, while the language used to describe these applications differs, the properties of these functions remain the same. In the context of weighted voting games, prime implicants are known as minimal winning coalitions; and in the AF tradition, the same essential points might be termed minimal deprivation bundles. To illustrate the unique characterization of identification functions via such minimal bundles, recall the examples of Table 1. According to rows 1(a)-1(c), a unit is classified as poor if they are deprived on any combination of four dimensions. Thus, as already hinted, the binary identification function can be represented as the union (disjunction) of all minimal bundles containing four elements. Written in sum-of-products form (used hereafter), which relies on modulus 2 arithmetic, this is:[6]

$$\begin{aligned} f_1 = &(d_1 \cdot d_2 \cdot d_3 \cdot d_4) + (d_1 \cdot d_2 \cdot d_3 \cdot d_5) \\ &+ (d_1 \cdot d_2 \cdot d_4 \cdot d_5) + (d_2 \cdot d_3 \cdot d_4 \cdot d_5) \end{aligned} \tag{2}$$

And the function in rows 2(a)-(c) of Table 1 can be uniquely characterised as:

$$f_2 = (d_1 \cdot d_2 \cdot d_4 \cdot d_5) + (d_1 \cdot d_3) + (d_4 \cdot d_3) + (d_5 \cdot d_3) \tag{3}$$

which contains minimal bundles of either two or four elements (dimensions). Evidently, these characterizations dispense with any reference to the particular (non-unique) weights or cut-off that define the true points of each function. Again, note each element (literal) in each minimal bundle is non-redundant and operates as a 'swing'. For instance, in row 2(a) the sum of weights on dimensions 1, 2, 4 and 5 is $0.54 > k = 0.515$; so, removing even the smallest weight ($w_2 = 0.047$) takes the weight sum below the specified cut-off. By this logic, longer bundles containing all the same elements as a minimal bundle (and more) are superfluous to the characterization since the minimal bundle is just a subset of a longer one. Thus, the set of minimal deprivation bundles (prime implicants of f) lists the shortest possible bundles of deprivations that are sufficient to identify a unit as poor.

Using representations of the form above, the Alkire-Foster poverty headcount can be directly estimated by evaluating the function f across the $i \in \{1, \ldots, n\}$ units of interest, each with a vector of deprivations, $D_i = (d_{i1}, \ldots, d_{im}) \in \mathcal{B}^m$. As before, we thus have:

$$h_i = f(D_i) \implies h = \mathrm{E}\left[f(D_i)\right] \approx \frac{1}{n} \sum_{i=1}^{n} f(D_i) \tag{4}$$

[6] The equivalent complete DNF representation is:

$$\begin{aligned} f_1 = &(d_1 \wedge d_2 \wedge d_3 \wedge d_4) \vee (d_1 \wedge d_2 \wedge d_3 \wedge d_5) \\ &\vee (d_1 \wedge d_2 \wedge d_4 \wedge d_5) \vee (d_2 \wedge d_3 \wedge d_4 \wedge d_5) \end{aligned}$$

Adopting the perspective of deprivation bundles, we see that the same headcount can be calculated by counting which rows of the deprivation matrix match to the members of the unique set of true points of the identification function:

$$H = \frac{1}{n} \sum_{X \in \mathcal{B}^m} f(X) \sum_{i=1}^{n} \min[X = D_i] \tag{5a}$$

$$= \frac{1}{n} \sum_{X \in \mathcal{T}} f(X) \sum_{i=1}^{n} \min[X = D_i] \tag{5b}$$

$$= \frac{1}{n} \sum_{X \in \mathcal{T}} \sum_{i=1}^{n} \min[X = D_i] \tag{5c}$$

$$= \frac{1}{n} \sum_{X \in \mathcal{T}^P} \sum_{i=1}^{n} \min[X \leq D_i] \tag{5d}$$

and where the final terms, which sum over i, count the units that match to specific rows of the truth table; and the minimum is evaluated element-wise. This indicates one may work directly with the truth table, using the proportion of sample observations, π, at each unique point of the Boolean threshold function. From this, the equivalent expression is:

$$H = \sum_{X \in \mathcal{T}} \pi(X) = \sum_{Y \in \mathcal{T}^P} \pi(Y) \; + \sum_{Z \in \mathcal{T}^{NP}} \pi(Z) \tag{6}$$

The simplicity (and computational efficiency) of this aggregate procedure can be grasped from Table A1 (Supplementary Material). It shows that the headcount associated with the functions represented by either of equations (2) or (3) is given by the inner product of the vector of sample proportions and the relevant vector of true points.

This section has clarified that Boolean threshold functions cannot be uniquely characterized by the weight vector and cut-off deployed. As such, comparing different functions in terms of their weights and/or cut-offs may be at best unhelpful (inefficient) and at worst misguided. In contrast, I have shown that minimal deprivation bundles (the Blake canonical form) provides a unique representation of the identification function. And, poverty headcounts can be calculated easily using the set of true points associated with the given function. Of course, this latter step appears superficially similar to the standard identification and counting procedure. The important difference here is that one uses the threshold function to identify the true points (and unique set of minimal bundles); and it is these points that are then matched to an aggregated version of the deprivation matrix, based on its unique rows (mirroring a truth table).[7] Further analytical advantages of approaching identification in this way become apparent below.

Lastly, it merits comment that whilst all poverty identification functions on the form of equation (1) can be

[7] This aggregation is not strictly necessary but is both intuitive and substantially reduces computation time (when looking across different functions).

represented by a unique collection of minimal bundles, not all feasible collections of bundles (implicants) can be represented as Boolean threshold functions. A reason is that the classification rules underlying any given collection may not admit a separating structure, meaning they cannot be expressed in terms of a specific set of weights and cut-off. This issue has been discussed elsewhere (Crama and Hammer, 2011) and somewhat limits the extent to which deprivation bundles may be relied upon exclusively to construct multi-dimensional poverty identification functions. Nonetheless, and as I show below, various approaches can be used to construct feasible threshold functions from proposed information about either admissible minimal bundles or the relative importance of different dimensions.

4 Applications

The remainder of this paper demonstrates some of the practical applications that derive from the main insight thus far – namely, that minimal bundles of deprivations provide a valuable and unique representation of counting-based multidimensional poverty identification functions. In doing so, and in order to place the ideas in a concrete setting, I draw on deprivation data from Mozambique, a low income country in south eastern sub-Saharan Africa. The country is of interest since it has achieved one of the world's most rapid and sustained rates of per capita economic growth since the end of conflict in 1992; however, recent consumption poverty estimates have raised concerns as to how well aggregate growth has translated into broad-based welfare gains (DNEAP, 2010; Arndt et al., 2012; DEEF, 2016). Indeed, the latest household survey from 2014/15 indicates that headcount consumption poverty affected 46% of the population versus 53% in 2002.[8] At the same time, the same surveys indicate more consistent gains in non-consumption dimensions, including ownership of assets and access to public goods, such as education services. This motivates a multi-dimensional analysis.

In order to proceed with a counting-based multi-dimensional poverty analysis, it is necessary to select the deprivation dimensions of interest. Although this decision can be somewhat controversial, feasible choices are often limited by the availability of consistent data over time, as well as exclusion of highly correlated dimensions. The dimensions selected for the present exercise are summarised in Table 2, which reports the share of households deprived in each of eight individual dimensions across four existing waves of nationally-representative household survey data collected by the national statistics agency in 1996/97, 2002/03, 2008/09 and 2014/15. The dimensions cover three broad areas: human capital (literacy of the household head); housing conditions (access to electricity, water and sanitation, and roofing quality); and ownership of assets (transport assets, information and communications assets, and durable goods). For each dimension, households who are classified as deprived receive a score of one and zero otherwise.[9]

[8] Survey weights are applied in all estimates in this section.
[9] Further details on the data and construction of the underlying deprivation dimensions is available on request from

Table 2: Dashboard of deprivations experienced by households, national means

	1996/97		2002/03		2008/09		2014/15	
	H	(s.e.)	H	(s.e.)	H	(s.e.)	H	(s.e.)
Literacy	47.75	(0.55)	45.57	(0.54)	44.65	(0.48)	41.57	(0.46)
Electricity	93.88	(0.26)	91.07	(0.31)	84.80	(0.35)	72.94	(0.42)
Clean water	72.99	(0.49)	58.56	(0.53)	57.55	(0.48)	47.66	(0.47)
Sanitation	95.55	(0.23)	85.97	(0.37)	82.00	(0.37)	71.63	(0.42)
Roofing	78.21	(0.46)	70.85	(0.49)	67.26	(0.45)	58.00	(0.46)
Transport	82.40	(0.42)	65.59	(0.51)	54.66	(0.48)	55.63	(0.46)
Information	63.10	(0.53)	42.80	(0.53)	37.45	(0.47)	24.64	(0.40)
Durables	87.29	(0.37)	79.48	(0.43)	68.70	(0.45)	49.65	(0.47)
Average	77.65	(0.46)	67.48	(0.50)	62.13	(0.47)	52.72	(0.47)

Notes: Cells indicate the share of households deprived on a given dimension; 'average' is the simple column-wise mean; standard errors in parentheses.
Source: author's estimates.

Table 2 indicates changes in well-being have been heterogeneous across dimensions. While we see progress in all dimensions over the full 18 year period, the pace of change is inconsistent. This implies that when constructing a multidimensional indicator, the relative importance attributed to different dimensions, or the specific combination of deprivations used to classify households as poor, is likely to matter. With this in mind, Table 3 estimates multidimensional poverty headcounts based on the same set of eight variables, applying three alternative identification functions chosen for illustrative purposes. The first, denoted f_e, contains all eight bundles of seven deprivations and thereby corresponds to an equal weight vector with cut-off $k = 7/8$. The second function, denoted f_u, represents an (extreme) alternative allowing for significant variation in the magnitudes of the dimensional weights. It was generated pseudo-randomly and also contains just eight minimal bundles, varying in length (i.e., the number of domains required to identify a unit as poor) from between 3 and 7 deprivations. The third function, denoted f_m, mimics the structure of the international MPI. Concretely it gives each of the three broad areas (human capital, housing conditions and asset ownership) an equal weight and, in turn, the dimensions making-up each area are equally weighted among themselves. Table A2 (Supplementary Material) sets out the dimensional weights and cut-offs employed as parameters in each of the three identification functions; and Tables A3 to A5 enumerate their respective minimal deprivation bundles.

All the estimates in Table 3 confirm strong progress in poverty reduction over time. Even so, the table also suggests that multidimensional rates of poverty remain high – e.g., in 2014/15 more than one in five households were deprived in at least seven of eight dimensions (function f_u); and, broadly speaking, at

the author (see also DEEF, 2016). For the exercise in Section 2, the same data is used but the first three dimensions are excluded.

Table 3: Estimates of multidimensional poverty headcount, alternative identification functions

	1996/97		2002/03		2008/09		2014/15	
f	H	(s.e.)	H	(s.e.)	H	(s.e.)	H	(s.e.)
Equal bundles, f_e	54.71	(0.55)	35.10	(0.51)	29.09	(0.44)	20.43	(0.38)
Unequal bundles, f_u	61.59	(0.54)	48.96	(0.54)	44.03	(0.48)	34.88	(0.45)
MPI-type bundles, f_m	61.71	(0.54)	48.87	(0.54)	44.83	(0.48)	35.31	(0.45)

Notes: Cells indicate the share of households (in %) classified as multidimensionally poor for alternative identification functions, as indicated (see text); standard errors in parentheses.
Source: author's estimates.

least one in three were deprived in two of the three broad areas used to replicate the structure of the MPI (f_m). The remainder of this section explores the features of these identification functions, comparing insights from the perspective of weights and cut-offs versus the perspective of (minimal) bundles.

4.1 Poverty decompositions

An immediate application of characterizing multi-dimensional poverty in terms of bundles is to decompose the contribution of each minimal deprivation bundle to the overall headcount. Using previous notation, the absolute contribution of minimal bundle $X \in \mathcal{T}^P$ to H, can be calculated as:

$$\tilde{H}^b(X) = \pi(X) \; + \sum_{Y \in \mathcal{T}^{NP}} \left(\frac{\min[X \leq Y]}{\sum_{Z \in \mathcal{T}^P} \min[Z \leq Y]} \right) \cdot \pi(Y) \tag{7}$$

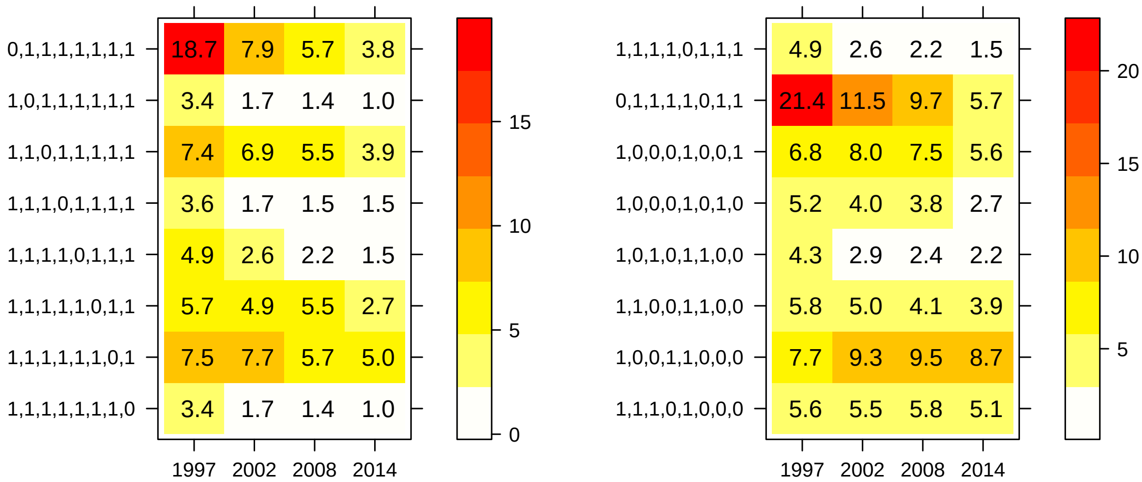
which is a direct modification of equation (6), the difference being an adjustment for the contribution of the non-prime implicants (deprivation bundles that are non-minimal). Indeed, while the prime implicants are unique, the non-prime implicants are likely to be absorbed by more than one prime implicant. Consequently, the contribution of minimal bundle X to H includes the sum of the contributions of each non-prime implicant absorbed by X (as identified by the numerator of the term in parentheses) divided by the number of prime implicants that absorb the same non-prime implicant (given by the denominator of the term in parentheses). So, effectively, the contribution of each non-prime implicant is allocated equally across those prime implicants by which it is absorbed.[10] This avoids multiple-counting and ensures: $H = \sum_{X \in \mathcal{T}^P} \tilde{H}^b(X)$.

What does this look like in practice? Figure 3 plots the above decomposition for the Mozambique data applying the two identification functions (f_e, f_u); and Figure A1 (Supplementary Material) does the same for the MPI-type function (f_m), which contains a munch longer set of minimal deprivation bundles. Both

[10] Equal weighting is not strictly required; nonetheless, the present decomposition is driven only by the properties of the relevant Boolean function, which ensures it is consistent across different contexts.

Figure 3: Decomposition by minimal deprivation bundles

(a) Equal length bundles, f_e (b) Unequal length bundles, f_u



Note: The figures depict the absolute contribution of each minimal deprivation bundle (denoted on the vertical axis) to the aggregate poverty headcount corresponding to a given identification function, (f_u and f_e), calculated per equation (7); survey starting years are depicted on the horizontal axis.
Source: author's calculations.

panels of the first figure indicate that in the earliest period a single bundle accounted for a relatively large share of the overall headcount; moreover, in both cases this bundle is highly similar – namely, containing all deprivations excluding literacy of the household head and ownership of some means of transport. The plots also indicate different paces of poverty reduction across bundles, even including increases in the (absolute) share in a few bundles of panel (b). This finding reveals a comparatively slower pace of improvement in the particular group of dimensions spanned by these bundles (e.g., literacy, sanitation and roofing), as well as some positive correlation among them.

The above decomposition points to further lines of analysis. One is to re-express (transform) the same decomposition to give the marginal contribution of each dimension to the headcount. To do so, I standardize the matrix of points representing all minimal bundles, such that each row sums to one and the individual cells indicate the proportional contribution of each variable (deprivation dimension) to the given bundle. Then, multiply the transpose of the earlier decomposition, represented in matrix form, by the standardized minimal bundles matrix, denoted $[B]$. That is:

$$\tilde{H}^d = [\tilde{H}^b]' \times [B] \tag{8}$$

Critically, neither this nor the earlier decomposition depends on the specific (i.e., non-unique) magnitudes of the weights or cut-off deployed to define the minimal deprivation bundles. Consequently, the present

Table 4: Dimensional decomposition of multidimensional poverty headcount

	(a) Equal, f_e		(b) Unequal, f_u		(c) MPI-type, f_m	
	AF	New	AF	New	AF	New
Literacy	2.38	2.38	10.52	8.65	12.74	6.42
Electricity	2.77	2.77	1.48	3.42	3.48	5.14
Clean water	2.35	2.35	1.10	2.99	2.58	3.74
Sanitation	2.70	2.70	2.89	4.05	3.29	4.84
Roofing	2.70	2.70	13.22	9.39	3.11	4.56
Transport	2.54	2.54	0.82	1.72	3.29	3.48
Information	2.21	2.21	2.81	2.06	2.40	2.44
Durables	2.78	2.78	2.47	3.03	3.97	4.25
Total	20.43	20.43	35.31	35.31	34.88	34.88

Notes: Cells indicate the absolute contribution of each deprivation dimension (in rows) to the overall multi-dimensional poverty headcount using alternative identification functions and decomposition procedures; the 'AF' decomposition procedure is the Alkire-Foster approach, as per equation (9); the 'new' procedure refers to the approach given by equation (8); equal, unequal and MPI-type bundles refer to the identification functions f_e, f_u and f_m, respectively enumerated in Tables A3 – A5.
Source: own estimates.

decompositions are unique to each identification function, which differs from the decompositions in the Alkire-Foster tradition where the contribution of deprivation dimension j is typically calculated as:

$$\tilde{H}(j) = w_j \mathrm{E}\left(h_i \times \frac{d_{ij}}{\sum_{j=1}^{m} d_{ij} w_j}\right) \tag{9}$$

Note this conventional decomposition approach effectively represents the share of units both classified as poor and deprived in a given dimension, multiplied by the weight attributed to that dimension, implying the magnitude of the weights play a direct role in the decomposition. Table 4 compares results from the procedure proposed here (denoted 'new'), as given by equation (8), and the conventional procedure (denoted, 'AF'), using the three identification functions presented above and focussing on data for the most recent period (2014/15). For the first identification function in column (a), which corresponds to the full set of deprivation bundles spanning seven dimensions (i.e., a vector of equal weights), there is no difference in the two decompositions. However, when more complex identification functions are used, as in columns (b) and (c), the two decompositions differ considerably (in absolute and relative terms) and do not even yield identical dimensional rankings. Absolute contribution differences are also large – e.g., in the case of the MPI-type function, the contribution of the literacy dimension falls by around half (from 12.74 to 6.42) when the new decomposition is applied. This clearly demonstrates some of the imprecisions that can arise when relying on non-unique weights to explore the properties of different identification functions.

The decompositions discussed in this section focus on the (short) set of minimal deprivation bundles. These are of substantive interest since each dimension in a bundle is non-redundant and thus necessary to identify a unit as poor. Supersets of these minimal bundles add dimensions that are not required to identify a unit as poor; thus, these dimensions are effectively given a weight (importance) of zero in the matrix standardization procedure that feeds into the dimensional decomposition. Nonetheless, the same decompositions may be applied using the full set of deprivation bundles that are sufficient to identify a unit as poor (all true points). This leads to an alternative dimensional decomposition, which is derived from treating each deprivation bundle as being of equal likelihood and, as such, demands a different interpretation (see further below). For further reference, Table A6 reports dimensional decompositions by survey year, based on both the minimal deprivation bundles (as above), as well as the full set of deprivation bundles, in each case using the identification function f_u.

4.2 Dimensional power analysis

The second application develops some of the ideas presented above and, in doing so, provides further intuition. As already noted, the weight vector applied in the Alkire-Foster identification procedure is conventionally interpreted as being directly informative of the relative importance of each dimension. The previous discussion suggests this is not likely to be correct, particularly since weights are often non-unique to the identification function. Instead, and as elaborated in the weighted voting game literature, formal measures of dimensional power (influence) can be derived from the true points of the threshold function and the set of prime implicants in particular (e.g., Lucas, 1983; Leech, 2002). Such formal measures of power not only focus on the unique features of a given function but also combine information about how the weights and cut-off underpinning a chosen function interact to give different dimensions more or less effective influence.

To illustrate this procedure, Table 5 compares the underlying weights used to generate the identification functions f_u and f_m, against four indexes of dimensional power calculated from the (true) points of the same functions.[11] The first measure is the normalized Banzhaf index (Dubey and Shapley, 1979), which is widely employed in the context of weighted voting games to indicate the influence of each voter under a particular set of rules (for further details on metrics of power and their interpretation see Lucas, 1983; Freixas and Gambarelli, 1997). It is calculated as the share of deprivation bundles in which a given dimension is a 'swing' (or pivot), which is simply equal to the number of minimal bundles in which a dimension appears, divided by the total number of dimensions appearing across all minimal bundles. Thus, the Banzhaf dimension power for f_u is simply the column shares in Table A4.

[11] The equal length bundles, function f_e, is not analysed here since the power and weight vectors are identical in this special case.

Table 5: Alternative metrics of dimensional power for f_u and f_m

	$\vec{w}$	Banzhaf	Banzhaf^w	Shapley	Nucleolus
(a) Unequal bundles, f_u:					
Literacy	0.297	0.206	0.237	0.339	0.286
Electricity	0.036	0.118	0.101	0.049	0.071
Clean water	0.036	0.118	0.101	0.049	0.071
Sanitation	0.074	0.088	0.080	0.058	0.071
Roofing	0.323	0.206	0.240	0.363	0.286
Transport	0.032	0.088	0.080	0.025	0.071
Information	0.127	0.088	0.080	0.058	0.071
Durables	0.074	0.088	0.080	0.058	0.071
(b) MPI-type bundles, f_m:					
Literacy	0.333	0.193	0.195	0.379	0.300
Electricity	0.083	0.120	0.119	0.086	0.100
Clean water	0.083	0.120	0.119	0.086	0.100
Sanitation	0.083	0.120	0.119	0.086	0.100
Roofing	0.083	0.120	0.119	0.086	0.100
Transport	0.111	0.108	0.109	0.093	0.100
Information	0.111	0.108	0.109	0.093	0.100
Durables	0.111	0.108	0.109	0.093	0.100

Notes: Cells indicate the relative influence of each deprivation dimension (in rows) based on the identification function represented in Figure 3(b); column $\vec{w}$ is the raw vector of weights; $\bar{B}$ reports the (dimension-wise) means of the standardized matrix of minimal deprivation bundles; Banzhaf and Shapley are as described in the text; all columns sum to one.
Source: author's estimates.

The second metric is a modified Banzhaf index that accounts for the different lengths of each minimal bundle and ascribes higher weights to shorter bundles (presuming they are more likely to be realized). It is derived directly from the standardized matrix of minimal bundles and reports the average weight of each dimension calculated across all minimal bundles, where the weight of a positive (non-zero) dimension within each bundle is the inverse of the total number of positive dimensions in the bundle and zero otherwise. So, this metric is just the weighted counterpart of the conventional Banzhaf metric. The third power measure is the Shapley value (Shapley, 1953), also frequently used in formal game theoretic analysis, which represents an estimate of what constitutes a fair distribution (representation) of the contribution of each dimension to all bundles sufficient to identify a unit as poor (its average marginal contribution). It is calculated from the characteristic function of the game defined by the Boolean threshold function, which amounts here to the list of points and the indicator whether they belong to the set of true points. Finally, the nucleolus is an alternative solution concept in transferable utility games indicating a stable division of hypothetical pay-offs that minimizes inequities between dimensions (players), and which also has been proposed as a relevant power metric (Montero, 2013).

Looking across the columns of Table 5, the exercise shows that although there seems to be a positive correlation between the vector of weights and the various power measures, there are also important differences. In the case of f_u, all four measures recognise that at least three of the dimensions (sanitation, information and durables) have identical power (relative importance), despite their unequal weights. Furthermore, the two Banzhaf power measures, calculated from the minimal bundles, suggest that the electricity and clean water dimensions have a greater influence (relative importance) than the sanitation dimension, despite receiving substantially lower weights. Put differently, the ratio of the weights taken from any two dimensions generally does not correspond to the ratio of the corresponding power indexes. Similar differences between the powers and weights emerge from the f_m function, suggesting that even in this more conventional case, differences in weights do not map directly to differences in relative importance. Indeed, under all power indexes excluding the Shapley value, the four housing conditions dimensions receive a power value that is greater than or equal to the values attributed to the three asset dimensions, despite the latter receiving larger weights.

In general, the power indexes tend to place less emphasis on differences between weights, often either allocating the same power values to multiple different weights or at least allocating values that are more similar in magnitude than the weights themselves. Even so, the power indexes evidently are not identical. The differences here correspond to the distinctive properties of each index or, effectively, different conceptualizations of what constitutes power (influence) in game theoretic settings (for elaboration see Laruelle, 1999). Furthermore, certain power indexes such as the nucleolus have been interpreted as a set of shadow prices that represent the equilibrium 'value' of each dimension under competitive bargaining (Montero, 2013). The point is that power metrics provide a rigorous means to determine the contribution of different dimensions to the outcomes of a given identification function, which cannot be determined directly from the weights (or cut-off). As such, these metrics can be used to evaluate how different dimensions trade-off against each other in identifying who is poor, which in part addresses the critique of Ravallion (2011).

To elaborate further on the distinction between dimensional weights and their game theoretic power, Figure A2(a) employs a set of random weights and cut-offs and plots the weight associated with each dimension (y-axis) against the Banzhaf power metric (x-axis) calculated for the same dimension from the given weight/cut-off vector. This immediately confirms that the magnitude of each dimension's weight is not always a good guide to its power – in fact, the pair-wise correlation between dimensional weights and Banzhaf power is approximately zero in this simulation. Figure A2(b) shows the corresponding histogram of the mean absolute difference between each *vector* of weights and the associated Banzhaf metrics of dimensional power, stated in logarithms. This shows that, on average, the absolute difference between dimensional weights and their Banzhaf power is over 0.4 log units or around 50%, supporting that one magnitude does not directly imply the other.

Lastly, the novel insights provided by an analysis of dimensional power are revealed by how measures of power vary with changes in the cut-off. Remaining with the example of function f_m, Figure 4(a) presents a conventional sensitivity analysis, showing different estimates of the poverty headcount for different values of k, keeping the weights fixed throughout, and where the original magnitude of the cut-off is indicated by the vertical line. What is of primary interest, however, is Figures 4(b)-(d), which illustrate how the Banhzaf powers for different dimensions also vary with k.[12] In contrast to plot (a), we find no smooth or even systematic relationship between changes in the cut-off and the dimensional powers. For instance, small variations either side of the original cut-off *both* induce an increase in the estimated power of the literacy and asset dimensions, but a decrease for all housing dimensions. Also, intuitively, all dimensional powers are equal when the cut-off is at its maximum. These findings reveal that dimensional weights do not provide a stable guide to the relative importance of different dimensions; in contrast, their importance materially depends on the particular value of the cut-off deployed for identification. This underlines the merit of using the unique characterization from minimal bundles in order to understand and further analyse the properties of different identification functions.

5 Conclusion

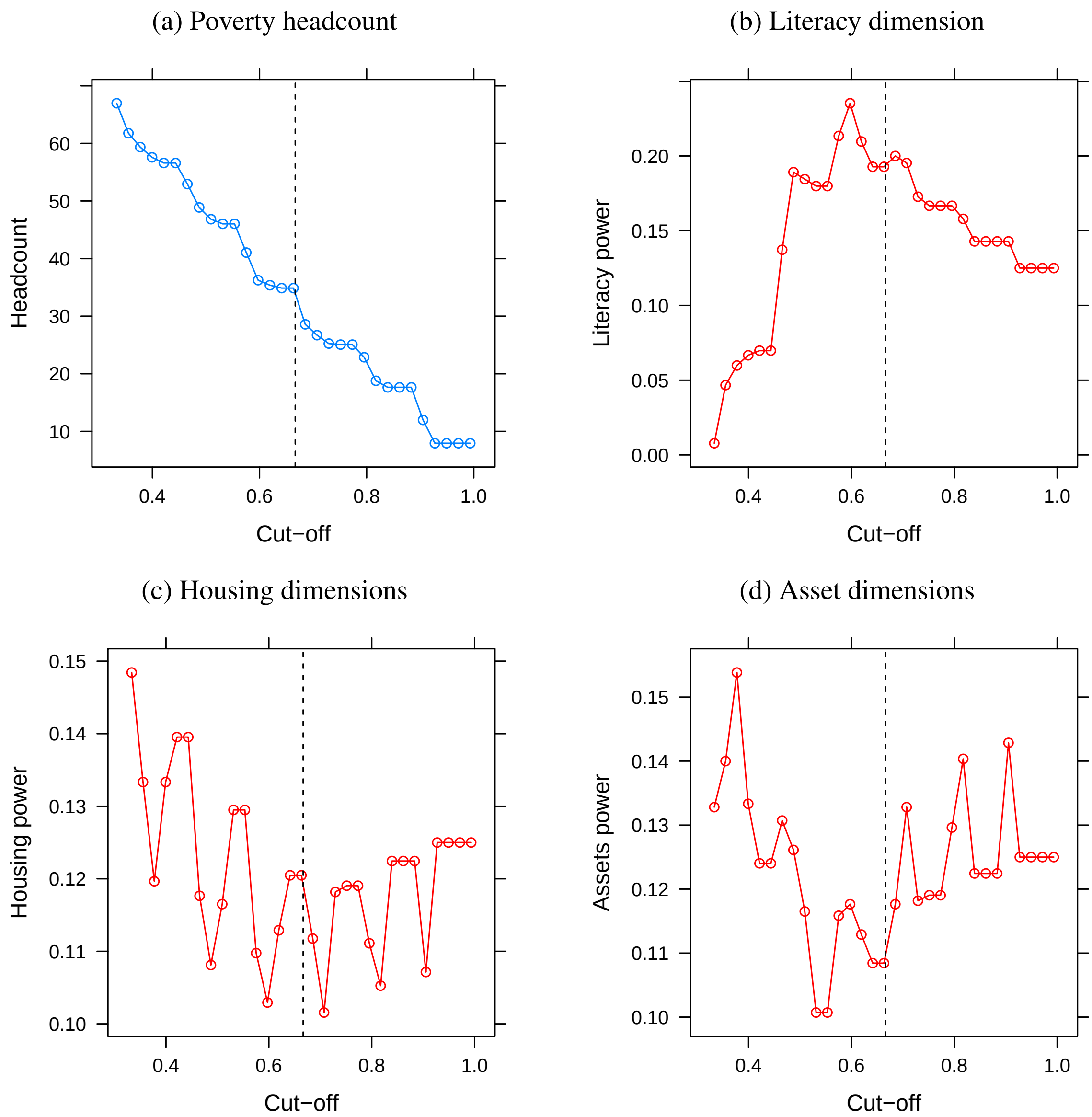
The purpose of this paper was to revisit the way in which identification functions in the Alkire-Foster tradition are understood. My point of departure was the observation that existing literature treats differences in the numerical values of the weight and cut-off parameters as a reliable and direct indication of differences in both who is identified as poor, as well as the relative importance of each dimension. Indeed, if this were not the case, there would be little to justify why these parameters are thought of as appropriate objects of public debate.

The main contribution of the present analysis was to show that both of the above conventional interpretations concerning the weight and cut-off parameters cannot be sustained. In light of the equivalence between the general form of the Alkire-Foster identification function and Boolean threshold functions, I showed that the weights and cut-offs are merely intermediate inputs that define the set of true points. Critically, due to the finite nature of the output space, an infinite number of weights and cut-offs will map to the same identification function. In turn, I demonstrated that identification functions can be uniquely characterised with respect to the set of minimum deprivation bundles, which is the analogue to minimum winning coalitions in the weighted voting literature.

The merit of considering identification in terms of minimum bundles, rather than the parameter inputs,

[12] Dimensions are aggregated into their groups since each dimension in each group receives the same weight.

Figure 4: Sensitivity to changes in the cut-off (Boolean threshold)



Note: Panel (a) plots the aggregate poverty headcount in Mozambique in 2014/15 for the MPI function (f_m), varying only the cut-off; panels (b)-(d) plot the Banzhaf powers corresponding to different groups of deprivation dimensions for alternative values of the cut-off.
Source: author's calculations.

was revealed via two applications illustrated with recourse to different types of identification functions, including a near replica of the international MPI. The first application proposed a new decomposition, showing both the absolute marginal contribution of each minimum bundle to the poverty headcount, as well as the contribution of each individual dimension. The advantage of the latter, relative to conventional approaches, is that the new decomposition does not depend on the vector of weights and, instead, is fixed for each unique identification function. Second, I illustrated how various formal measures of (voting) power, taken from game theory, can be applied to evaluate the relative importance of different dimensions. Moreover, some of these metrics can be interpreted as dimensional shadow values and thus rigorously capture how dimensions are implicitly traded-off against one another in any given function.

In sum, the approach set out here provides a productive and complementary set of tools to analyse poverty identification functions. It also points to some directions for further research. For example, either minimal deprivation bundles or metrics of power might be used as an intuitive basis for the design of identification functions. In the former case, linear programming methods can be used to identify whether a set of weights and a cut-off can be derived from a proposed (incomplete) set of true and false points. In the latter case, a technical challenge is how best to derive a stable mapping from proposed bundles or powers to a particular vector of weights and cut-offs. Second, the application of Boolean theory suggests progress may be made in evaluating the robustness of poverty comparisons to differences in identification functions, not via direct changes to weights or the cut-off but rather to the minimal bundles themselves.

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Appendix: Boolean threshold functions

Definition 5.1. A Boolean function on m variables is a function on $\mathcal{B}^m$ into $\mathcal{B}$, where $\mathcal{B} = \{0, 1\}$, and $\mathcal{B}^m$ is the m-fold cartesian product $\{0, 1\}^m$.

This definition indicates the logical nature of Boolean functions, namely that they map a vector of m binary variable inputs (known as literals) into a scalar that takes the value of either zero or one, which refer to (Boolean) FALSE and TRUE outputs respectively. Thus:

Definition 5.2. A point $X = (x_1, x_2, \ldots, x_m)$ is a true point of the Boolean function f if $f(X) = 1$; respectively, a false point obtains if $f(X) = 0$. In turn, $\mathcal{T}(f)$ denotes the set of true points; and the set of false points is denoted $\mathcal{F}(f)$.

Definition 5.3. A Boolean function f in m variables is a positive threshold function if there exist m positive weights, $\vec{w} \in \mathbb{R}^m_+$, and a threshold or cut-off $0 < k \leq \sum_{j=1}^m \omega_j$, such that for any point X:

$$f(X) = \begin{cases} 1 & \text{if } \sum_{j=1}^m w_j x_j \geq k \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Whilst somewhat trivial, the above definitions highlight that a given Boolean function effectively partitions the space of feasible points, given by all 2^m possible combinations of variable inputs (literals), into members of the true or false point sets. And the weights and cut-off used in Boolean threshold functions merely constitute the partitioning mechanism, as illustrated by the truth table set out in Appendix Table A1 (see above). Also, since both the input and output space of a Boolean function is finite, the number of different feasible Boolean threshold functions in m variables must be finite, taking strict upper bound 2^{2^m}. (Since at least Dedekind, counting the number of different kinds of Boolean functions remains an active area of research).[13] Put differently, while $(\vec{w}; k)$ can take an infinite number of values (being reals), some different choices of $(\vec{w}; k)$ will be consistent with the same Boolean threshold function.

Given the above, it is sensible to look at the output vector in order to characterize Boolean functions. Indeed, as Definition 5.2 suggests, the set of true points can uniquely distinguish between different positive Boolean functions:

Definition 5.4. Two Boolean functions f and g are said to be identical when their associated truth tables are identical. Namely: if, $\forall X \in \mathcal{B}^m : f(X) = g(X)$ then $\mathcal{T}(f) = \mathcal{T}(g) \Leftrightarrow f = g$.

The drawback of using truth tables to characterise a Boolean (threshold) function is that, in all but the most simple instances, the number of true points ($\#\mathcal{T}$) can be large. Nonetheless, further properties of Boolean functions indicate that even complex functions can be given a shorter and unique summary representation. To see this, some further definitions are required.

Definition 5.5. A minterm on $\mathcal{B}^m$ is an elementary conjunction containing exactly m literals, which takes the value one (TRUE) at a unique point in the input space. So, given a Boolean function f evaluated at point X, if $f(X) = 1$, then the corresponding minterm is: $\phi_f(X) = (\bigwedge_{j|x_j=1} x_j \bigwedge_{k|x_k=0} \bar{x_k}) = 1$, where $\bar{x_k} = 1 - x_k$ indicates the complement of x_k.

[13] Muroga et al. (1970) finds that the number of unique positive Boolean threshold functions in 5 variables is just 119, but increases exponentially with the number of variables, being already more than 2 million for 8 variables.

Definition 5.6. A Boolean function f is positive (and monotonic) if for two points X and Y in m variables , $f(X) \geq f(Y)$ whenever $\forall j \in \{1, \ldots, m\} : x_j \geq y_j$.

From Definition 5.5, it follows that the disjunction (union) of all minterms that define the true points of a function f provides a complete representation of the same function. This is known as the minterm disjunctive normal form (DNF) of f. Taking this further, where f is positive, then minterms will often encode redundant information. And since Definition 5.6 indicates that positive Boolean functions cannot be switched to FALSE by switching any individual literal from zero to one, it must be the case that any minterm of a positive function that defines a true point can be represented *without* reference to any variables that enter in complemented form. That is, if f is a positive function, then:

$$\forall\, X \in \mathcal{T}(f) : \underbrace{\bigwedge_{j|x_j=1} x_j \bigwedge_{k|x_k=0} \bar{x_k}}_{\phi_f(X)} = \underbrace{\bigwedge_{j|x_j=1} x_j}_{\phi_f^+(X)} = 1 \tag{11}$$

Without need for proof, this proposition implies the standard result that every positive Boolean function can be represented by a DNF containing only positive implicants (denoted, ϕ_f^+).

The final step is to note that at least some positive implicants can be superfluous to the representation of positive monotonic Boolean threshold functions. To see this, again consider two true points X and Y of a positive function f, meaning $f(X) = f(Y) = 1$; so, if $\forall j : x_j \leq y_j$, then the elementary conjunction of positive points in X provides a shorter but nonetheless sufficient representation of the truth value of *both* points X and Y, implying $\phi^+(Y)$ is not required to represent the function. Thus:

Definition 5.7. An implicant of a Boolean function f is said to be non-prime if it is absorbed by at least one other implicant of f. Thus, the set of non-prime implicants corresponds to a proper subset of true points: $\mathcal{T}^{NP} = \{Y \in \mathcal{T}(f) \mid \exists X \in \mathcal{T}(f) \backslash Y : \min_{\forall j}(x_j \leq y_j) = 1\} \subset \mathcal{T}$.[14] And the set of prime implicants contains those not absorbed by any other implicant, corresponding to the set of points: $\mathcal{T}^P = \mathcal{T} \backslash \mathcal{T}^{NP}$.

[14] Equivalently, we say implicant $\phi^+(X)$ absorbs implicant $\phi^+(Y)$ if the set of indices pertaining to positive literals in X is a proper subset of the indices pertaining to positive literals in Y – i.e., define the set of indices $M = \{1, ..., m\}$; and, the set of indices pertaining to positive literals of point X, $M^X = \{j \in M : x_j = 1\}$. So, X absorbs $Y \iff M^X \subset M^Y$.

A Supplementary material

Table A1: Enumeration of bundles in five dimensions

	Bundles						Row 1			Row 2		
b	d_1	d_2	d_3	d_4	d_5	π	(a)	(b)	(c)	(a)	(b)	(c)
1	0	0	0	0	0	0.089	0	0	0	0	0	0
2	1	0	0	0	0	0.082	0	0	0	0	0	0
3	0	1	0	0	0	0.021	0	0	0	0	0	0
4	1	1	0	0	0	0.125	0	0	0	0	0	0
5	0	0	1	0	0	0.107	0	0	0	0	0	0
6	1	0	1	0	0	0.048	0	0	0	1	1	1
7	0	1	1	0	0	0.006	0	0	0	0	0	0
8	1	1	1	0	0	0.024	0	0	0	1	1	1
9	0	0	0	1	0	0.000	0	0	0	0	0	0
10	1	0	0	1	0	0.000	0	0	0	0	0	0
11	0	1	0	1	0	0.000	0	0	0	0	0	0
12	1	1	0	1	0	0.000	0	0	0	0	0	0
13	0	0	1	1	0	0.000	0	0	0	1	1	1
14	1	0	1	1	0	0.000	0	0	0	1	1	1
15	0	1	1	1	0	0.000	0	0	0	1	1	1
16	1	1	1	1	0	0.000	1	1	1	1	1	1
17	0	0	0	0	1	0.001	0	0	0	0	0	0
18	1	0	0	0	1	0.008	0	0	0	0	0	0
19	0	1	0	0	1	0.006	0	0	0	0	0	0
20	1	1	0	0	1	0.053	0	0	0	0	0	0
21	0	0	1	0	1	0.017	0	0	0	1	1	1
22	1	0	1	0	1	0.039	0	0	0	1	1	1
23	0	1	1	0	1	0.012	0	0	0	1	1	1
24	1	1	1	0	1	0.115	1	1	1	1	1	1
25	0	0	0	1	1	0.000	0	0	0	0	0	0
26	1	0	0	1	1	0.007	0	0	0	0	0	0
27	0	1	0	1	1	0.003	0	0	0	0	0	0
28	1	1	0	1	1	0.047	1	1	1	1	1	1
29	0	0	1	1	1	0.005	0	0	0	1	1	1
30	1	0	1	1	1	0.017	1	1	1	1	1	1
31	0	1	1	1	1	0.016	1	1	1	1	1	1
32	1	1	1	1	1	0.151	1	1	1	1	1	1

Table A2: Dimension-specific weights and cut-offs for alternative identification functions

Dimension	f_e	f_m	f_u
Literacy	0.125	0.333	0.297
Electricity	0.125	0.083	0.036
Clean water	0.125	0.083	0.036
Sanitation	0.125	0.083	0.074
Roofing	0.125	0.083	0.323
Transport	0.125	0.111	0.032
Information	0.125	0.111	0.127
Durables	0.125	0.111	0.074
k	0.875	0.667	0.668

Notes: Against each deprivation dimension, cells indicate the respective weights used for identification; k is the cut-off; column names indicate the identification function – namely: f_e is the equal weights function, f_m is the MPI-type function, and f_u is the unequal weights function.
Source: author's estimates.

Table A3: Minimal deprivation bundles for function f_e (equal length bundles)

id.	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	Sum
128	1	1	1	1	1	1	1	0	7
192	1	1	1	1	1	1	0	1	7
224	1	1	1	1	1	0	1	1	7
240	1	1	1	1	0	1	1	1	7
248	1	1	1	0	1	1	1	1	7
252	1	1	0	1	1	1	1	1	7
254	1	0	1	1	1	1	1	1	7
255	0	1	1	1	1	1	1	1	7
Sum	7	7	7	7	7	7	7	7	56

Notes: 'id.' is the bundle identifier (row number); columns $d_1, \ldots, d_8$ indicate the deprivation dimensions, listed in the same order as in Table 2; 'Sum' is the row or column count of necessary deprivations.
Source: author's estimates.

Table A4: Minimal deprivation bundles for function f_u (unequal length bundles)

id.	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	Sum
24	1	1	1	0	1	0	0	0	4
26	1	0	0	1	1	0	0	0	3
52	1	1	0	0	1	1	0	0	4
54	1	0	1	0	1	1	0	0	4
82	1	0	0	0	1	0	1	0	3
146	1	0	0	0	1	0	0	1	3
223	0	1	1	1	1	0	1	1	6
240	1	1	1	1	0	1	1	1	7
Sum	7	4	4	3	7	3	3	3	34

Notes: 'id.' is the bundle identifier (row number); columns $d_1, \ldots, d_8$ indicate the deprivation dimensions, listed in the same order as in Table 2; 'Sum' is the row or column count of necessary deprivations.
Source: author's estimates.

Table A5: Minimal deprivation bundles for function f_m (MPI-type bundles)

id.	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	Sum
32	1	1	1	1	1	0	0	0	5
48	1	1	1	1	0	1	0	0	5
56	1	1	1	0	1	1	0	0	5
60	1	1	0	1	1	1	0	0	5
62	1	0	1	1	1	1	0	0	5
80	1	1	1	1	0	0	1	0	5
88	1	1	1	0	1	0	1	0	5
92	1	1	0	1	1	0	1	0	5
94	1	0	1	1	1	0	1	0	5
104	1	1	1	0	0	1	1	0	5
108	1	1	0	1	0	1	1	0	5
110	1	0	1	1	0	1	1	0	5
116	1	1	0	0	1	1	1	0	5
118	1	0	1	0	1	1	1	0	5
122	1	0	0	1	1	1	1	0	5
144	1	1	1	1	0	0	0	1	5
152	1	1	1	0	1	0	0	1	5
156	1	1	0	1	1	0	0	1	5
158	1	0	1	1	1	0	0	1	5
168	1	1	1	0	0	1	0	1	5
172	1	1	0	1	0	1	0	1	5
174	1	0	1	1	0	1	0	1	5
180	1	1	0	0	1	1	0	1	5
182	1	0	1	0	1	1	0	1	5
186	1	0	0	1	1	1	0	1	5
200	1	1	1	0	0	0	1	1	5
204	1	1	0	1	0	0	1	1	5
206	1	0	1	1	0	0	1	1	5
212	1	1	0	0	1	0	1	1	5
214	1	0	1	0	1	0	1	1	5
218	1	0	0	1	1	0	1	1	5
226	1	0	0	0	0	1	1	1	4
255	0	1	1	1	1	1	1	1	7
Sum	32	20	20	20	20	18	18	18	166

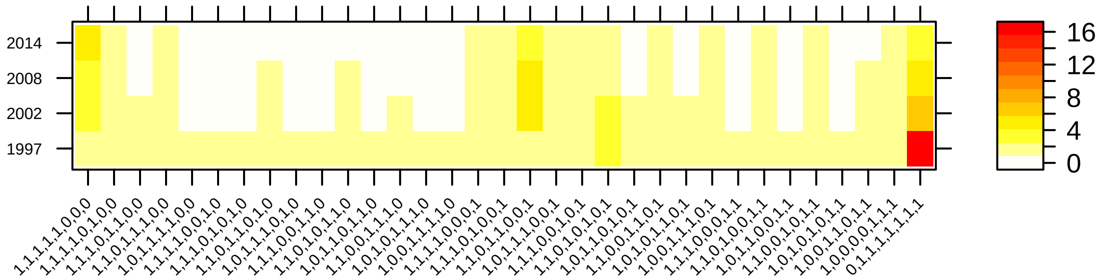
Notes: 'id.' is the bundle identifier (row number); columns $d_1, \ldots, d_8$ indicate the deprivation dimensions, listed in the same order as in Table 2; 'Sum' is the row or column count of necessary deprivations.
Source: author's estimates.

Table A6: Decomposition of multidimensional poverty headcount, function f_u

	Minimal deprivation bundles				All deprivation bundles			
	1996/97	2002/03	2008/09	2014/15	1996/97	2002/03	2008/09	2014/15
Literacy	11.18	10.84	10.29	8.65	6.00	5.86	5.69	4.96
Electricity	7.12	4.92	4.38	3.42	8.56	7.24	6.77	5.49
Clean water	6.76	4.39	3.96	2.99	7.53	5.42	5.09	3.89
Sanitation	6.83	5.39	5.07	4.05	8.52	7.17	6.68	5.19
Roofing	14.06	12.38	11.59	9.39	8.36	7.10	6.71	5.55
Transport	3.23	2.36	1.93	1.72	7.26	4.74	3.76	3.25
Information	5.99	3.63	3.19	2.06	7.24	4.59	4.11	2.72
Durables	6.55	4.96	4.41	3.03	8.25	6.76	6.03	4.26
Total	61.71	48.87	44.83	35.31	61.71	48.87	44.83	35.31

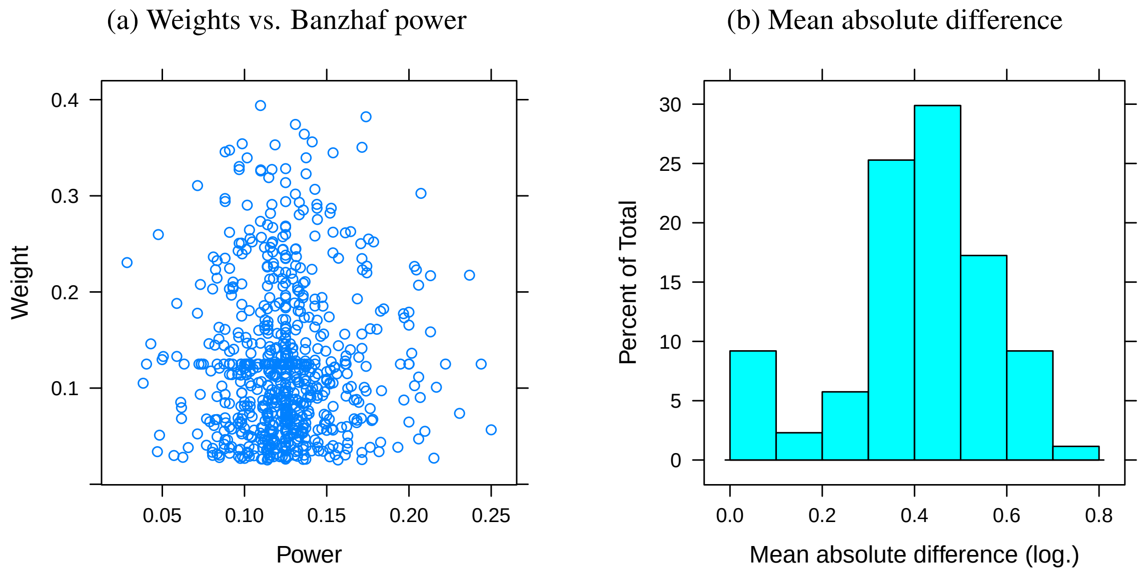
Notes: Cells indicate the absolute contribution of each deprivation dimension (in rows) to the multi-dimensional poverty headcount using alternative decomposition procedures, namely that based on the set of minimal deprivation bundles and that based on all deprivation bundles that are true points of the function f_u.
Source: author's estimates.

Figure A1: Decomposition by minimal deprivation bundles, function f_m (MPI-type)



Note: The figure depicts the absolute contribution of each minimal deprivation bundle (denoted on the horizontal axis) to the aggregate poverty headcount corresponding to the f_m identification function; survey starting years are depicted on the vertical axis.
Source: author's calculations.

Figure A2: Comparison of random dimensional weights and powers



Note: Panel (a) plots randomly-drawn dimensional weights against their Banzhaf powers; panel (b) plots the mean log absolute difference between random vectors of weights and their Banzhaf powers; cut-off vector is also drawn randomly, not shown.
Source: author's calculations.