Inflation and its policy in an unequal economy

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Abstract

This paper investigates the possible role of inequality in macroeconomic outcomes and how monetary policy may be influenced by redistribution. Policy effectiveness is expected to be dependent on the response of the agents in the economy, therefore we model heterogeneity. We model growth in an unequal society while capturing the role of money and its policy. The intuition is that inflation, through money growth, crowds out savings, reduces capital stock accumulation and therefore reduces growth. In an unequal society, inflation extends the inequality gap as those with fewer resources have less room to adjust their savings-consumption decision around inflation fluctuations. We adopt an overlapping generation model and money is introduced through a Cash in Advance constraint and we specifically model a heterogeneous economy to build a framework that portrays and explains growth, inflation and inequality dynamics. Analytical findings show that inequality is explained by inflation and money growth while capital accumulation relates to distribution disparities. A numerical analysis of the model simulating the South African economy shows that inflation and inequality are positively related and changes in monetary policy through inflation targeting reduce the level of inequality. The impact of policy is also noted to be better under equality for macroeconomic aggregates such as consumption, output and welfare. Particularly welfare reflects negative relationship with inequality and but a positive one the level of inflation suggesting a trade off. The findings are expected to assist policy formulation to allow achievement of sustainable stability in economies.

Keywords: inflation, redistribution, policy, inequality, money
JEL CLASSIFICATIONS: D31, E13, E5, O41, O42

1 Introduction

Although inevitable, inequality is a challenge when it hinders decent living. Despite the downward trend in global inequality, reducing inequality remains a priority as outlined in the Sustainable Development Goals (UN General Assembly, 2015). Policy makers seek to address the vast inequality worldwide and central banks have developed a growing interest in the relationship between inequality and monetary policy. This interest arose following the 2007-8 financial crisis and the advent of new approaches to monetary policy. Some studies preceding the financial crisis, although inconclusive (Lo, 2012), attribute the crisis to a rise in inequality. This is explained by the increase in housing prices which resulted in a credit boom (Rajan & Lines, 2010; Kumhof et al., 2015). Inequality has the potential to influence the effectiveness of monetary policy as the degree of income and wealth holding of agents determines responses to policy. Monetary policy can also have redistributive effects affecting income and wealth inequality. We therefore investigate the distributive effects of monetary policy as well as the feedback of inequality on policy efficiency for growth attainment.

Should inequality affect how monetary policy is thought of? A debate exists with regards to the role of monetary policy in addressing issues of inequality. One school of thought supports that inequality can be reduced through expansionary policy as it boosts the economy and this may benefit the poor by creating opportunities for them to enter into economic activities. A second view point is of the opposite perception, suggesting that benefits are better aligned to the initially wealthy and this widens the distributional gap as the rich are advantaged unfairly from the beginning (Coibion et al., 2017). One last point of view, expressed by Bernanke (2015) expresses that "Monetary policy is neutral" and therefore has a limited long-run effect on inequality. Traditionally the relationship between inequality and monetary policy has been secondary for central banks. However departing from tradition, a line of questioning attempting to understand the linkage between the distributional structure of economies and macroeconomic policies has arisen (Bullard et al., 2014). Focus is on policies whose formulation have not been directly associated with inequality such as monetary policy, which we intend to delve in.

The main objective of this study is to investigate the bidirectional relationship between inequality and monetary policy. We focus on how inequality dynamics relate to macroeconomic aggregates and how monetary policy plays a role through an inflation tax mechanism. We also consider the link between inequality and monetary policy by examining the effect of inequality on the effectiveness of monetary policy. The expectation is that households adopt policy differently based on their heterogeneous characteristics, therefore affecting policy success.

The current framework adopts a Diamond (1965) type of overlapping generations model in which individuals live over two periods. We closely follow Yanagihara and Lu (2013) and Crettez et al. (1999, 2002) but specifically assume that households are heterogeneous allowing the model to portray inequality dynamics. Individuals differ in initial capital endowment, giving rise to inequality which persists due to idiosyncratic ability. Through a Hahn and Solow (1995) type of Cash in advance constraint on consumption, we introduce money which therefore channels inflation.

The model analysis shows that capital and distributional dynamics relate to inflation. Inflation taxes income and this negatively affects the level of capital accumulation as inflation lessens the funds available for the savings-consumption decision. Increase in inflation erodes household purchasing power disproportionately for lower income households that are cash dependent (Erosa & Ventura, 2002). Inequality dynamics persist in equilibrium as a result of idiosyncratic ability shocks and affect the capital invested into production. Analytical evaluation suggest that the existence of inflation and inequality have negative effects on the macroeconomic outcome of output and consumption.

In a numerical analysis in which we calibrate our model to match the South African and the sensitivity analysis confirms the relationship between inequality and inflation. South Africa is characterised by inequality and is notably one of the most unequal economies in the world (World Bank, 2019). According to the World Bank (2019), persistence in both income and wealth inequality, is attributed to non-inclusive growth policies and to low inter-generational mobility. In a non-inflationary economy, although inequality effect remains negative, its reduction leads to a flourished economy as compared to an inflationary state. We also note that inflation fuels inequality while it is detrimental for the accumulation of capital stock. An economy in which inequality exists is shown to have less favorable effects on the effectiveness of monetary policy. Findings also show that welfare improves in both a non-inflationary and equal economy. Bidirectional causality between inequality and monetary policy is noted given the better performance of the macroeconomic indicators of consumption and output within the target inflation in an equal economy.

The growth-inequality literature is largely inconclusive with studies finding varying outcomes. Various channels have been suggested to explain the relationship stretching from socio-political economy (Benhabib & Rustichini, 1996), to credit market imperfections (Benabou, 1996, 2002). This line of analysis relates to Loury (1981), Getachew and Turnovsky (2020) and Basu and Getachew (2020) in which market imperfections relating to credit are modelled in order to understand the relationship. Studies have also considered investment related to physical and human capital and more recently fertility (De La Croix & Doepke, 2003). The area remains open to deeper understanding and findings. We contribute to this literature by studying the growth-inequality nexus through an inflation and monetary policy mechanism.

The strand of literature that addresses policy efficiency exists and the norm in addressing inequality has been mainly through fiscal policy instruments such as taxes (Galor & Zeira, 1993). Conformable studies that seek to understand the growth and inequality also identify channels through which various public policies affect the equality-efficiency trade-off (Getachew & Turnovsky, 2015, 2020; Getachew, 2010, 2012, 2016; Turnovsky, 2015). There is also further analysis on the efficiency-equality trade off that extends to welfare and García-Peñalosa and Turnovsky (2007) find that these policies reduce welfare inequality. The findings generally suggest a positive relationship between inequality and growth given the public policies meant to improve growth (Chatterjee & Turnovsky, 2012). These studies all contributed to the growth and inequality relationship but with a main focus on only fiscal policy. We extend this evaluation of policy in relation to inequality by focusing on monetary policy whose mandate excludes redistribution.

The importance of money and inflation for dynamics of inequality is particularly of interest given the question of the role of monetary policy in addressing inequality. The importance of money for the dynamics of inequality is also widely documented. Controlling inflation is one explicit objective of monetary policy but the effects of such policy on redistribution should be of relevance. The intuition is that curbing inflation lowers inequality as high inflation is expected to affect the low income households more and therefore exacerbating the inequality gap (Easterly & Fischer, 2001). Ghossoub and Reed (2017) and Albanesi (2007) find that as the poor's income is weakened by inflation, so is their bargaining power in the fiscal policy decision making. Given the question surrounding the importance of inflation in addressing inequality, inflation is expected to affect the labour supply decision. Bulir (1998) highlights how inflation affects agents on different redistribution levels along the spectrum. We extend this to understand the feedback of inequality of monetary policy efficiency.

Another strand of literature on the role of inflation in the growth of an economy is a significant feature in our study. In order to find the role of money in the growth analysis studies such as Dotsey and Sarte (2000); Jin (2009); Gupta and Stander (2018) include a Cash-in-Advance (CIA) constraint. When the constraint on money holdings applies to consumption, inflation has the effect of a stochastic tax. The popular notion is that inflation acts as a tax that induces a substitution between goods consumption and leisure (Gillman et al., 2001). Our study sets up the growth-inflation relationship in an unequal economy.

This study aims to model the inequality-inflation-policy nexus. A couple of studies have taken the direction of evaluating the relationship while including growth (Galli, 2001; Ma, 2019; Doepke & Schneider, 2006). We build on the intuition that given a growth model, the capital –labor ratio is affected by changes in the nominal interest, through the introduction of a cash-in-advance constraint which in-turn affects inequality given the disparities in income holding of agents. Chu and Cozzi (2014)and and Kakar (2014) introduces money through a CIA constraint. There is a reduction in consumption inequality as all agents are affected despite their position in on the redistribution spectrum. Disposable income and wealth inequality also reduces in margin.

In the present study the aim is to contribute to explaining the mechanism of money (inflation) on inequality and welfare trade-off. We carry out an investigation of how inequality-growth-inflation relate by introducing heterogeneity in agents. This builds from incomplete market models Aiyagari (1994); Krusell and Smith (1998); Huggett (1993); Broer et al. (2020). Given existing growth models the objective is to understand the role of money on the growth factors namely capital (private/public) and human capital. We also study the effects of heterogeneous agents on the dynamics of redistribution and success of macroeconomic policy. This contributes to the existing body of literature around the issue of macroeconomic policy, with particular focus on monetary policy and the global goal of reducing inequality. The study seeks to assist possibly assist future policy formulation to be simultaneously pro-stability, pro-growth and pro-equality.

The paper is organised such that Section 2 outlines the model. Sections 3 and 4 presents the dynamics and the steady state of the macroeconomic aggregates which are affected by inflation in an unequal economy. Section 5 describes model calibration and presents numerical analysis of the model. Section 6 concludes the paper.

2 Model

Our investigation utilises an overlapping generations model (OLG) characterised by a two - period lifetime in the economy with a continuum of heterogeneous households, $i \in (0,1)$. An individual is modelled to live over two periods of time,one in which they are young and active and another when they are old. In period t an individual i chooses consumption and the savings that maximise his utility. An individual enters his old age in $(t+1)$ in which he invests his savings. The consumption decision is meant to satisfy the individual's lifetime and the savings decision will also influence money holding in order to consume in $t+1$ as a fraction of consumption is required to be financed by money. The central bank as government representative creates the money in the economy which it then be injected into households. The individuals are assumed to be identical over the periods of time and there is no growth in population of N. Producing firms are owned by all the households who then supply labour as an inelastic input of production utilising a Cobb Douglas function of Ak_{it}^{α} with constant returns to scale. The model is built based on an overlapping generation framework (OGM) following the Diamond (1965) style and money demand is introduced through a Cash in Advance constraint (CIA). The extension of including money demand follows a Hahn and Solow (1995) type of Cash in advance constraint as in Crettez et al. (1999, 2002)
Therefore in this study the representation of a real economy will be displayed through changes of individuals behavior captured over two different phases of their life.

2.1 The Individuals

The economic activities of the young in t, the decision stage, determine an individuals lifespan. An individual inelastically supplies labour for an income which they allocate between money holding and savings for old age consumption. The utility of the individuals born in time t is an individuals lifetime utility derived through maximisation of consumption in youthful period and in old age where utility is given below.
$U(c_{it}, c_{it+1}^{o}) = U(c_{it}) + \beta U(c_{it+1}^{o})$

where $U(c_{it}), U(c_{it+1}^{o})$ are logarithmic utility functions and consumption at young age and in old age is denoted by c_{it} and c_{it+1}^{o}, respectively.
In the overlapping generations model (OGM), individuals live for a finite period of time. Over a two period lifespan, one is economically active in the first period and then retires in the final period of life. Focus is placed on period t where the generation born in this period plans out for its young and older state. Therefore, an individual seeks to maximise utility over their lifetime as follows:

$$\max U(c_{it}, c_{it+1}^{o}) \tag{1}$$

subject to the budget constraints:

$$w_{it} + \tau_t = c_{it} + s_{it} + \frac{M_{it}}{P_t} \tag{2}$$

$$c^o_{it+1} = R_{it+1}s_{it} + \frac{M_{it}}{P_{t+1}} \tag{3}$$

where τ_t is government lump-sum transfers, w_{it} denotes real wage rate from l_{it} labour, s_{it} is real savings and R_{it+1} is return on investment or savings. We assume the individual has a unit of labour, $l_{it} = 1$. Current money holding in real terms is denoted by $\frac{M_{it}}{P_{it}}$ where M_{it} is the money demanded by an individual in period t and P_t is the price level in period t which can also be simplified to m_{it}. Next period money holdings in real terms are $\frac{M_{it}}{P_{t+1}}$, which is also expressed as $\frac{m_{it}}{1+\pi_{t+1}}$ where π_t is inflation given by $\frac{P_t}{P_{t-1}} - 1$.

Cash in Advance

The Cash in Advance (CIA) constraint suggests that an individuals requires some cash holding in order to transact, which in this case is to enable consumption in old age. This is given by

$$M_{it} \geq \theta c^o_{it+1} P_{t+1} \tag{4}$$

where $\theta \in (0, 1)$ is a fraction of old age consumption that is financed by money holding or cash (Hahn & Solow, 1995; Crettez et al., 2002).
It is only a fraction because there are returns on savings that finance consumption in old age which is the only expenditure (Yanagihara & Lu, 2013).This is so because if all savings are kept as money holding, capital accumulation would be eliminated (Crettez et al., 1999) but as it is necessary for growth Hahn and Solow (1995) suggest the constraint in (5).

The constraint is binding if the rate of return on capital (R_{t+1}) remains greater than the rate of return on money holdings (P_t/P_{t+1}). When the value of return on non-monetary assets is greater than that on monetary ones then money is regarded to be of no value. Value is held in the asset with higher rate of return and therefore agents will not hold money beyond what is inferred by the cash in advance constraint. We assume this is true and that the constraint is binding for all generations.

The CIA constraint becomes

$$\frac{M_{it}}{P_{t+1}} = \frac{m_{it}}{1 + \pi_{t+1}} = \theta c^o_{it+1} \tag{5}$$

Optimal choice of the Individual

The optimal choices for an individual are based on both the consumption budget and CIA constraints therefore these are combined. Making s_{it} the subject of formula in (3) gives an expression of savings

$$s_{it} = \left(c^o_{it+1} - \frac{m_{it}}{1+\pi_t}\right)\frac{1}{R_{it+1}} \tag{6}$$

and substituting it into (2) we obtain the combined budget as

$$c_{it} = w_{it} + \tau_t - \left(c^o_{it+1} - \frac{M_{it}}{P_{t+1}}\right)\frac{1}{R_{it+1}} - \frac{M_{it}}{P_t} \tag{7}$$

Furnished with the combined budgetary conditions as well as the CIA constraint (5), we solve for the household problem using the inter-temporal budget equation

$$c_{it} = w_{it} + \tau_t - \Big[\frac{(1-\theta)}{R_{it+1}} + \theta(1+\pi_{t+1})\Big]c^o_{it+1} \tag{8}$$

where $(1+\pi_{t+1}) \equiv \dfrac{P_{t+1}}{P_t}$

Substituting for c_{it} using (8) in the maximisation function we obtain

$$\max_{c^o_{it+1}} \left\{ \ln\left[w_{it} + \tau_t - \Big[\frac{(1-\theta)}{R_{it+1}} + \theta(1+\pi_{t+1})\Big]c^o_{it+1}\right] + \beta \ln c^o_{it+1} \right\} \tag{9}$$

and maximise (9) subject to (8) to obtain the optimal choice of individual consumption and savings as

$$\frac{c^o_{it+1}}{c_{it}} = \frac{\beta R_{it+1}}{1-\theta+\theta(1+\pi_{t+1})R_{it+1}} \tag{10}$$

This shows that the marginal rate of substitution between current and future consumption is the discounted returns on savings less costs of holding money as cash. The need to hold money is shown to crowd out savings and therefore a fraction of returns from investing savings is lost. Inflation, which is the cost of holding cash adjusts the return on capital received by the agent. Using the inter-temporal budget constraint (8) we substitute for c_{it} into (10) and obtain old age consumption as

$$c^o_{it+1} = \Big(\frac{\beta}{1+\beta}\Big)\Big[\frac{R_{it+1}}{1-\theta+\theta(1+\pi_{t+1})R_{it+1}}\Big](w_{it}+\tau_t) \tag{11}$$

Given c^o_{it+1}, the savings equation (6) becomes;

$$s_{it} = \Big(\frac{\beta}{1+\beta}\Big)(w_{it}+\tau_t)\Big(\frac{1-\theta}{1-\theta+\theta(1+\pi_{t+1})R_{it+1}}\Big) \tag{12}$$

Savings in the economy are dependent on income flows which are based on wages and inflation adjusted return on capital and the lump sum transfers τ_t. As part of the income meant to finance next period consumption, the fraction necessary for cash holding also impacts the savings level.

2.2 Production

In this model firms are owned by all households in the economy. The young individuals provide labour while the old own capital stock. A Cobb-Douglas production function, Ak_{it}^{α} with constant returns to scale, is applied. The firm maximises profit in a perfectly competitive economy where resource prices, w_{it} and R_{it} for labour and capital respectively, are given.

The source of inequality in the model is initial capital endowment and an i.i.d idiosyncratic productivity shock (ϵ_{it}) which drives marginal productivity. This difference in productivity allows for persistence of inequality. The productivity shock follows a log linear distribution $\ln \epsilon_{it} \sim N(\frac{\nu^2}{2}; \nu^2)$. The production function in the economy is given by $Y_t = A\epsilon_{it}K_t^{\alpha}(N_t l_{it})^{1-\alpha}$ where Y_t, K_t and N_t are output,physical capital and population size respectively. Maximisation for individual firms yields

$$R_{it} = A\alpha\epsilon_{it}k_{it}^{\alpha-1} \tag{13a}$$

$$w_{it} = A\epsilon_{it}(1-\alpha)k_{it}^{\alpha} \tag{13b}$$

as the optimal price of capital and labour, where k_{it} and y_{it} are per capita capital stock and output respectively.

2.3 Government

The Central bank represents the government as a monetary authority that creates money at a rate η and provides a supply of money M_t^s in a period such that the growth of money supply is as follows:

$$M_{t+1}^s - M_t^s = \eta M_t^s \tag{14a}$$

$$M_{t+1}^s = (1+\eta)M_t^s \tag{14b}$$

The rate of money creation is therefore the difference between money supplied over two consecutive periods. The government utilises the money created to finance lump sum transfers τ_t which aid with consumption for individuals in their old age.

$$P_t\tau_t = \eta M_t^s; \tag{15}$$

Where τ_t is the transfer which we later express in terms of capital accumulation which is the driver of dynamics in the economy.

3 Dynamics of the economy in Equilibrium

In this section we show the equilibrium dynamics of key aggregates in the economy. Equilibrium in the economy can be achieved when the capital and money markets clear. We begin by showing market clearing conditions in period t and then aggregate the economy to obtain dynamics of physical capital and inequality and the steady state equilibrium.

Physical Capital Market

The condition for equilibrium in the capital market is that capital accumulation is the same as savings for each individual.

$$k_{it+1} = s_{it} \tag{16}$$

This holds assuming full depreciation of capital, such that the capital market equilibrium is achieved when aggregate capital stock in the next period equals current savings.

Money Market in equilibrium

Money market equilibrium is achieved under the condition that money supply and it's growth equals money demand as in (17) below

$$M_t^s = NM_{it} \tag{17}$$

Utilising the condition of money market in (17) and considering the definition of M_{it} from the binding CIA constraint in (5) we obtain

$$M_t^s = NM_{it} = NP_{t+1}\theta c_{it+1}^o \tag{18}$$

expressing money holdings.

Combining the old age budget constraint (3) and CIA (5) gives us current period old age individual consumption as;

$$c_{it+1}^o = \frac{1}{1-\theta} R_{it+1} s_{it} \tag{19}$$

which is used in (18) such that

$$M_t^s = NP_{t+1}\theta c_{it+1}^o \tag{20}$$

$$M_t^s = NP_{t+1}\frac{\theta}{1-\theta} R_{it+1} s_{it} = NM_{it} \tag{21}$$

which is the money market equilibrium of the economy in nominal terms.
Applying the money supply equation (14b) and physical capital equilibrium conditions (16) we substitute accordingly in (21). We divide by P_t to obtain the money market equilibrium in real terms. Using (13a) we replace for the real return on capital, R_{it} .

$$NM_{it} = M_t^s = (1+\eta)M_{t-1}^s$$

$$m_{it} = (1+\eta)\frac{\theta}{1-\theta}A\alpha\epsilon_{it}k_{it}^{\alpha-1}k_{it}$$

$$m_{it} = (1+\eta)\frac{\theta}{1-\theta}A\alpha\epsilon_{it}k_{it}^{\alpha} \tag{22}$$

This gives the real money holdings of a young individual in period t expressed in terms of capital stock. Individual money demand is shown to be dependent on the rate of money growth η, fraction of old age consumption, θ, as well as capital investment.

The government lump sum transfer established in (15) can then also be expressed in terms of physical capital terms by substituting for interest rate and s_{it} using equations (13a) and (16) respectively.

$$P_t\tau_t = \eta P_{t+1}\frac{\theta}{1-\theta}R_{it}s_{it}$$

$$\tau_t = \frac{A\alpha\theta}{1-\theta}\eta\epsilon_{it+1}k_{it+1}^{\alpha}(1+\pi_{t+1}) \tag{23}$$

Transfers are related to the rate of money creation which allows the government to make the lump sum distributions. They also increases in level of old age consumption and capital input investment.

Combining the transfer and money demand function yields the inflation in equilibrium which is a function of the growth of money and capital dynamics.

$$1+\pi_{t'+1} = (1+\eta)\frac{\epsilon_{it}k_{it}^{\alpha}}{\epsilon_{it+1}k_{it+1}^{\alpha}}$$

The savings function in (12) can therefore be expressed as:

$$k_{it+1} = \Big(\frac{\beta}{1+\beta}\Big)\big(A\epsilon_{it}(1-\alpha)k_{it}^{\alpha}+\tau_t\big)\Big(\frac{1-\theta}{1-\theta+\theta(1+\pi_{t+1})A\epsilon_{it+1}\alpha k_{it+1}^{\alpha-1}}\Big) \tag{24}$$

and substituting for τ_t using (23) yields

$$k_{it+1} = \frac{\beta A\epsilon_{it}\alpha k_{it^{\alpha}}}{1+\beta}\Big[\frac{(1-\alpha)(1-\theta)+\theta\eta}{1-\theta+\theta(1+\pi_{t+1})A\epsilon_{it+1}\alpha k_{it+1}^{\alpha-1}}\Big] \tag{25}$$

Capital stock accumulation is therefore determined by current physical capital investment and inflation. This suggest that inflation has a crowding effect on capital.

3.1 Aggregate Capital

In order to derive the aggregate capital stock we redefine (3) as in Appendix A such that the capital function becomes

$$\Upsilon k_{it+1} + (1+\pi_{t+1})X\epsilon_{it+1}k_{it+1}^{\alpha} = Z\epsilon_{it}k_{it}^{\alpha} \tag{26}$$

The log-normality assumptions :

$$\ln k_{it} \sim N(\mu_t; \sigma_t^2);$$
$$\ln \epsilon_{it} \sim N(\frac{\nu^2}{2}; \nu^2)$$

aid aggregation as detailed in Appendix A. The aggregate capital dynamics are shown as

$$\Upsilon k_{t+1} + (1+\pi_{t+1})Xk_{t+1}^{\alpha}e^{0.5\alpha(\alpha-1)\sigma_{t+1}^2} = Zk_t^{\alpha}e^{0.5\alpha(\alpha-1)\sigma_t^2} \tag{27}$$

which is an expression of the evolution of capital stock in the economy. Capital stock dynamics relate to future inflation and are a result of variation in capital given initial capital endowment.

Distributional dynamics

Inequality is determined by taking the variance of the capital equation (26): as detailed in Appendix B. Dynamics of inequality are therefore given by:

$$\sigma_t^2 = \frac{1}{\alpha^2}\ln\Psi(k_{t+1}, \pi_t, \sigma_{t+1}^2) \tag{28}$$

where $\Psi(k,\pi,\sigma 2) = \dfrac{\Upsilon^2 k_{t+1}^2 + X^2(1+\pi_{t+1})^2 k_{t+1}^{2\alpha}e^{\alpha^2\sigma_{t+1}^2+\nu^2} + 2X\Upsilon(1+\pi_{t+1})k_{t+1}^{1+\alpha}e^{0.5\alpha(\alpha+1)\sigma_{t+1}^2}}{[(\Upsilon^2 k_{t+1}^2 + X^2(1+\pi_{t+1})^2 k_{t+1}^{2\alpha}e^{\alpha(\alpha-1)\sigma_{t+1}^2} + 2X\Upsilon(1+\pi_{t+1})k_{t+1}^{1+\alpha}e^{0.5\alpha(\alpha-1)\sigma_{t+1}^2})]e^{\nu^2}}$

Inequality dynamics persist due to idiosyncratic ability shocks and have an impact on capital employed in production and the policy parameter of future inflation.
The evolution of inequality also highlights a relationship between future capital stock and inflation; a fraction of capital that is invested into production is affected by inflation.

4 Steady State

4.1 Capital and Inequality in the Steady State

In this section we outline the long run features of the economy through capital stock and inequality. In the steady state
$k_t = k_{t+1} = k$;

$\sigma_t^2 = \sigma_{t+1}^2 = \sigma^2$ and
$\pi_t = \pi_{t+1} = \pi$

conditions should hold given constant growth in variables. Applying these conditions to (27) and (28) we obtain steady state growth path of capital and distributional dynamics are as follows:

Capital:

$$k = \left[e^{0.5\alpha(\alpha-1)\sigma^2} \left(\frac{Z - (1+\pi)X}{\Upsilon} \right) \right]^{\frac{1}{1-\alpha}} \tag{29}$$

where Υ, X and Z are defined in Appendix (A). In the long run,capital is a function of inflation and initial capital disparities.

Inequality:

$$\sigma^2 = \frac{1}{\alpha^2} \ln \Psi^* \tag{30}$$

where $\Psi^*(k^*, \pi^*, \sigma^2*)$ is the steady state reduction from Ψ given in section 3.1.

Inflation

$$\pi = \eta \tag{31}$$

In the steady state the rate of inflation rate is equal to the money growth rate.

4.2 Steady State Aggregate Welfare, Output and Consumption

In this section we continue to outline the steady state path of other key macroeconomic aggregates in the economy.

Steady State Welfare

The steady state welfare is given below as shown in Appendix C

$$W \approx \frac{1+R}{R} \left(2\ln\omega - \ln\beta - \nu^2 + 2\ln k^\alpha + \sigma^2\alpha(\alpha-1) - \ln\left[\frac{s_2}{s_1}(u+1) \left(\frac{u^2}{(u+1)^2}(e^{\sigma^2} - 1) + 1 \right)^{-0.5} \right] \right) \tag{32}$$

Where

$u = \frac{s_2}{s_1} k^{1-\alpha} e^{0.5\alpha(\alpha-1)\sigma,k}$
$s_2 = \Upsilon(A\alpha)^{-1}$
$s_1 = \theta(1+\pi)$

Long run welfare is affected by the rate of inflation, distribution of capital and the variation of disparities in productivity. Ones position on the distribution spectrum determines welfare and it is key to note that inflation has a welfare reducing effect.

Steady State Output

We aggregate the Cobb -Douglas production function, $y_{it} = A\epsilon_{it}k_{it}^{\alpha}$ to obtain

$$y_t = Ak_t^{\alpha} e^{0.5\alpha(\alpha-1)\sigma_{t,k}}$$

as the aggregate output produced in the economy. In the steady state output is:

$$y = Ak^{\alpha} e^{0.5\alpha(\alpha-1)\sigma^2} \tag{33}$$

a function of both capital stock and its distribution.

Steady State Consumption

Total consumption for an individual is obtained by adding consumption across generations of life such that **C** = $c_{it} + c_{it+1}^{o}$.
Using $\ln c_t = \ln E(c_{it}) - 0.5\sigma_c^2$ and $lnc_{t+1}^{o} = \ln E(c_{it+1}^{o}) - 0.5\sigma_{c_{t+1}}^2$ we obtain c_t and c_{t+1}^{o} as derived in Appendix D and combine them such that

$$\mathbf{C} = \omega k_t^{\alpha} e^{-(0.5\alpha\sigma_{k,t}^2+\nu^2)} \left[\frac{1}{\beta} + \frac{s_2}{s_1^2}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}^2}-1)+1 \right) \right] \tag{34}$$

In the steady state consumption becomes:

$$\mathbf{C} = \omega k^{\alpha} e^{-(0.5\alpha\sigma_k^2+\nu^2)} \left[\frac{1}{\beta} + \frac{s_2}{s_1^2}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_k^2}-1)+1 \right) \right] \tag{35}$$

Steady state consumption is shown to be a function of inflation rate, capital accumulation and both capital stock and productivity inequality.

5 Numerical Analysis

A numerical analysis follows to continue evaluation of the inflation-inequality nexus. In order to do that we carry out a quantitative analysis of the effects of inflation on capital and inequality dynamics. We also look at how changes in inflation and inequality affect output, consumption and welfare in the economy. Parameters that are reflective of a feasible real economy are compiled with a focus around South Africa.

5.1 Calibration

Table 1 below details the parameter values set for the benchmark economy that we seek to analyse numerically. A psychological discount factor of 0.9975 is taken from Du Plessis, Smit, Steinbach, et al. (2014) and set as $\beta = 0.99^{30} \approx 0.74$ for a 30 day period (De La Croix & Michel, 2002, p255). Baseline value for capital share of capital (α) is obtained from Du Plessis et al. (2014), a study that focuses on South Africa. The rate of money growth is calibrated based on the inflation target (Heer & Maußner, 2011) which in the South African ranges between $3 - 6\%$ (shown in Appendix E). Long run capital stock is calibrated using data from the Federal Reserve Bank of St. Louis Dataset based on the capital-output ratio. The initial inequality is taken from the World bank overview statistics for South Africa in which the average Gini coefficient for 2015 is highlighted. The Total Factor Productivity (TFP), is normalised to 1 for simplification as it is not important to the analysis. R is set 30% and is used to obtain the social planner's discount factor measuring the importance placed on future generation's utility. The value of R is chosen implying a generational discount factor discount factor $(1 + R)^{-1}$, of 0.77 (Getachew & Turnovsky, 2020).

The numerical analysis exercise results are shown in this section. Given the South African economy parameters in the steady state and the initial values for capital stock and inequality based on the capital-output ratio data and the high inequality characteristic of the country respectively, an analysis of the model is carried out to obtain steady state inequality and capital stock in the modelled economy.

Table 1: Benchmark values

Baseline parametres		
capital share of income	α	0.23
initial inequality(variance)	σ^2	0.63
initial capital stock	k	4
preference discount factor	β	0.74
rate of money growth	η	0.1255-0.2665
fraction of old age	θ	0.3
total factor production	A	1
welfare discount factor	R	0.3
productivity shock/ability	ν^2	0.4

In order to analyse the dynamics of the model a selection of parameters are altered. Varying the policy parameter, target inflation, shows the role played by inflation on the accumulation of capital and extent of inequality. South Africa is one country that uses a range to express the inflation that the Central Bank strives to achieve and maintain while addressing issues of economic progression. The target inflation range for the country is between $3-6\%$ and is used in this study to calibrate for the rate of money growth, η.

Table 2: Capital stock and Inequality dynamics with policy adjustment

	Capital Stock	Inequality	Total consumption	Output	Welfare
Baseline					
target inflation at 3%	0.1378	0.2609	1.7500	0.6194	-12.7449
target inflation at 6%	0.1333	0.2660	1.4642	0.6144	-12.7921
Adjustments					
↓target inflation to 0%	0.1431	0.2550	2.1308	0.6252	-12.7043
↓target inflation to 2%	0.1395	0.2590	1.8650	0.6215	-12.7305
↑target inflation to 8%	0.1309	0.2687	1.3144	0.6117	-12.8259
↑target inflation to 10%	0.1290	0.2708	1.1909	0.6096	-12.8602

In Table 2, the lower limit of 3% displays a capital accumulation improvement and inequality reduction while the opposite is true for the upper limit. This persists with reduced inflation target. Inflation relates negatively to capital stock accumulation which can be explained in the model by the aspect that high inflation erodes the value of incomes and therefore reduces saving ability. This can also be noted in the way total output is henceforth lower as the rate of inflation hikes. Total consumption, C, is therefore also inversely related to inflation as agents find a balance between saving and consuming on lower valued income. Inflation also increases disparities in income showing the redistributional effects of inflation.

An economy with no money creation, and therefore no policy intervention is indicated by zero target inflation. In this case inequality reduces and the economic indicators including welfare. This corresponds with propositions in the analytical section which postulate that zero inflation is not the advisable target or stance for the Central Bank.

In Table 3 we consider how macroeconomic aggregates respond to changes in distribution in both an inflationary and non-inflationary economy. Inequality in the model stems from shock to productivity which highlights individual differences in ability. Increasing and lowering this shock shows that a larger variation in productivity, (ν^2) not only leads to increased capital inequality but it also affects capital stock accumulation negatively in the presence of inflation as shown in Table 3.

Table 3 also shows that the general absence of inflation improves capital accumulation and the inequality dynamics are reduced. The effect of inequality remains the same but in a non-

Table 3: Inequality in inflationary and non-inflationary economy

inflationary economy	Target inflation	Capital Stock	Inequality	Total consumption	Output	Welfare
+10% (ν^2)	3%	0.1374	0.2861	1.7134	0.6176	-12.8002
	6%	0.1329	0.2917	1.4320	0.6126	-12.8502
-10% (ν^2)	3%	0.1382	0.2356	1.7876	0.6212	-12.6895
	6%	0.1337	0.2402	1.4973	0.6162	-12.7339
non-inflation economy						
+10% (ν^2)	0%	0.1427	0.2796	2.0886	0.6234	-12.7569
-10% (ν^2)	0%	0.1435	0.2301	2.1740	0.6269	-12.6515

inflationary economy the inequality is lower and the capital accumulation is greater in margin. Welfare is better with low ability disparities and monetary policy in place. When there is no policy intervention however, inequality deteriorates welfare overall although lessening inequality improves it.

The total consumption and output produced is low in an inflationary economy in comparison to the non-inflationary economy, with consumption significantly different highlighting the effect of money particularly in the old age consumption decision. Higher productivity inequality results in lower output and consumption and welfare is low. The non inflationary economy yields similar results. However the welfare of the economy decreases with increased productivity inequality.

Table 4: Policy changes in an economy with equality

	Capital Stock	Total consumption	Output	Welfare
target inflation at 3%	0.1420	2.1772	0.6383	-12.1863
target inflation at 6%	0.1374	1.8428	0.6335	-12.2056

Finally, in Table 4 the effects of policy changes in an inflationary but equal society are explored. In this case ν^2 is set at zero. In comparison to results in Table 3, where inequality exists, the results show that an equal society is most favourable for macroeconomic aggregates and general welfare.

6 Conclusion

Debates around the effectiveness of monetary policy have arisen over time especially following the recent global financial crisis. This study endeavours to assess inflation and associated policy in an unequal society.

The analysis is carried out initially by modelling an economy in an overlapping generation framework with money introduced through cash in advance constraint. In the model inequality stems from differences in productive ability and the mechanism through which monetary policy through inflation may relate to inequality. The introduction of money allows for inflation build up which is controlled by rate of money growth. The model is solved analytically and findings suggest that the interaction of monetary policy, through inflation, and inequality exists. The steady state of the economy shows that capital, a driver of dynamics in the economy, is a function of inflation while inequality is also determined by inflation. The capital stock channel of relating monetary policy given inflation and inequality which can be linked to the asset pricing channel is highlighted in the study. The channel brings into perspective how capital as an input in production can be either accumulated (savings) for return or not given the household consumption decision which requires holding money instead.

To provide a quantitative analysis we carry out a numerical exercise. Calibration of the South African economy is carried out and the findings show that the relationship between inflation and inequality is conceivable although it may appear small in magnitude. Adjusting the policy parameter, through inflation targeting, shows that policy improves distribution disparities and capital stock. Redistribution is also more pronounced when the inequality shock is applied in a non-inflationary economy versus an inflationary one. It can also be noted that policy is less effective in improving macroeconomic variables in an unequal society. The results show that there is a complex relationship between monetary policy and redistribution which is tentative with regards to welfare alleviation.

The analysis shows that feedback exists surrounding macroeconomic variables relating to the livelihood of economic agents, their policies and equality or lack thereof. This then highlights the need for consideration of heterogeneity of economic agents in policy formulation despite the common line of thinking that monetary policy has no mandate to address inequality. The existence of a relationship between the two suggests that the efficiency of policy may be deterred by the differences in distribution of the economy. This study is however a simplified reflection of how economies work and is therefore room to extend the characteristics of the economy and its sectors for further analysis.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, *109*(3), 659–684.

Albanesi, S. (2007). Inflation and inequality. *Journal of monetary Economics*, *54*(4), 1088–1114.

Basu, P., & Getachew, Y. (2020). Redistributive innovation policy, inequality, and efficiency. *Journal of Public Economic Theory*, *22*(3), 532–554.

Benabou, R. (1996). Inequality and growth. *NBER macroeconomics annual*, *11*, 11–74.

Benabou, R. (2002). Tax and education policy in a heterogeneous-agent economy: What levels of redistribution maximize growth and efficiency? *Econometrica*, *70*(2), 481–517.

Benhabib, J., & Rustichini, A. (1996). Social conflict and growth. *Journal of economic growth*, *1*(1), 125–142.

Bernanke, S. B. (2015, June). *Monetary policy and iequality.* Brookings. Retrieved from https://www.brookings.edu/blog/ben-bernanke/2015/06/01/monetary-policy-and-inequality/

Broer, T., Harbo Hansen, N.-J., Krusell, P., & Öberg, E. (2020). The new keynesian transmission mechanism: A heterogeneous-agent perspective. *The Review of Economic Studies*, *87*(1), 77–101.

Bulir, M. A. (1998). *Income inequality: does inflation matter?* International Monetary Fund.

Bullard, J. B., et al. (2014). *Income inequality and monetary policy: a framework with answers to three questions* (Tech. Rep.).

Chatterjee, S., & Turnovsky, S. J. (2012). Infrastructure and inequality. *European Economic Review*, *56*(8), 1730–1745.

Chu, A. C., & Cozzi, G. (2014). R&d and economic growth in a cash-in-advance economy. *International Economic Review*, *55*(2), 507–524.

Coibion, O., Gorodnichenko, Y., Kueng, L., & Silvia, J. (2017). Innocent bystanders? monetary policy and inequality. *Journal of Monetary Economics*, *88*, 70–89.

Crettez, B., Michel, P., & Wigniolle, B. (1999). Cash-in-advance constraints in the diamond overlapping generations model: neutrality and optimality of monetary policies. *Oxford Economic Papers*, *51*(3), 431–452.

Crettez, B., Michel, P., & Wigniolle, B. (2002). Optimal monetary policy, taxes, and public debt in an intertemporal equilibrium. *Journal of Public Economic Theory*, *4*(3), 299–316.

De La Croix, D., & Doepke, M. (2003). Inequality and growth: why differential fertility matters. *American Economic Review*, *93*(4), 1091–1113.

De La Croix, D., & Michel, P. (2002). *A theory of economic growth: dynamics and policy in overlapping generations*. Cambridge University Press.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review*, *55*(5), 1126–1150.

Doepke, M., & Schneider, M. (2006). *Inflation as a redistribution shock: effects on aggregates and welfare.* National Bureau of Economic Research Cambridge, Mass., USA.

Dotsey, M., & Sarte, P. D. (2000). Inflation uncertainty and growth in a cash-in-advance economy. *Journal of Monetary Economics*, *45*(3), 631–655.

Du Plessis, S., Smit, B., Steinbach, R., et al. (2014). *A mediumsized open economy dsge model of south africa* (Tech. Rep.).

Easterly, W., & Fischer, S. (2001). Inflation and the poor. *Journal of money, credit and banking*, 160–178.

Erosa, A., & Ventura, G. (2002). On inflation as a regressive consumption tax. *Journal of Monetary Economics*, *49*(4), 761–795.

Galli, R. (2001). Is inflation bad for income inequality: The importance of the initial rate of inflation.

Galor, O., & Zeira, J. (1993). Income distribution and macroeconomics. *The review of economic studies*, *60*(1), 35–52.

García-Peñalosa, C., & Turnovsky, S. J. (2007). Growth, income inequality, and fiscal policy: What are the relevant trade-offs? *Journal of Money, Credit and Banking*, *39*(2-3), 369–394.

Getachew, Y. Y. (2010). Public capital and distributional dynamics in a two-sector growth model. *Journal of Macroeconomics*, *32*(2), 606–616.

Getachew, Y. Y. (2012). Distributional effects of public policy choices. *Economics Letters*, *115*(1), 56–59.

Getachew, Y. Y. (2016). Credit constraints, growth and inequality dynamics. *Economic Modelling*, *54*, 364–376.

Getachew, Y. Y., & Turnovsky, S. J. (2015). Productive government spending and its consequences for the growth–inequality tradeoff. *Research in Economics*, *69*(4), 621–640.

Getachew, Y. Y., & Turnovsky, S. J. (2020). Redistribution, inequality, and efficiency with credit constraints: Implications for south africa. *Economic Modelling*, *93*, 259–277.

Ghossoub, E. A., & Reed, R. R. (2017). Financial development, income inequality, and the redistributive effects of monetary policy. *Journal of Development Economics*, *126*, 167–189.

Gillman, M., Harris, M. N., & Matyas, L. (2001). Inflation and growth: Some theory and evidence.

Gupta, R., & Stander, L. (2018). Endogenous fluctuations in an endogenous growth model: An analysis of inflation targeting as a policy. *The Quarterly Review of Economics and Finance*, *69*, 1–8.

Hahn, F., & Solow, F. (1995). Perfectly flexible wages. *A critical essay on modern macroeconomic theory*.

Heer, B., & Maußner, A. (2011). The cash-in-advance constraint in monetary growth models.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of economic Dynamics and Control*, *17*(5-6), 953–969.

Jin, Y. (2009). A note on inflation, economic growth, and income inequality. *Macroeconomic Dynamics*, *13*(1), 138–147. doi: 10.1017/S1365100508070491

Kakar, V. (2014). On the redistributional effects of long-run inflation in a cash-in-advance economy.

Krusell, P., & Smith, A. A., Jr. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of political Economy*, *106*(5), 867–896.

Kumhof, M., Rancière, R., & Winant, P. (2015). Inequality, leverage, and crises. *American Economic Review*, *105*(3), 1217–45.

Lo, A. W. (2012). Reading about the financial crisis: A twenty-one-book review. *Journal of economic literature*, *50*(1), 151–78.

Loury, G. C. (1981). Intergenerational transfers and the distribution of earnings. *Econometrica: Journal of the Econometric Society*, 843–867.

Ma, E. (2019). Monetary policy and inequality: How does one affect the other? *Available at SSRN 3488931*.

Rajan, R. G., & Lines, F. (2010). How hidden fractures still threaten the world economy. *Princeton, New Jersey*.

Turnovsky, S. J. (2015). Economic growth and inequality: The role of public investment. *Journal of Economic Dynamics and Control*, *61*, 204–221.

Yanagihara, M., & Lu, C. (2013). Cash-in-advance constraint, optimal monetary policy, and human capital accumulation. *Research in Economics*, *67*(3), 278–288.

A Appendix

Appendix A: Aggregating Capital

Let:

$$\Upsilon = (1-\theta);$$

$$X = A\alpha\theta(1-\frac{\beta\eta}{1+\beta});$$

$$Z = \frac{A\beta(1-\alpha)(1-\theta)}{1+\beta}$$

Given a lognormality assumption :

$$k_{it} \sim N(\mu_t;\sigma_t^2)$$

$$E[(k_{it})^x] = k_{it}^x e^{0.5x(x-1)\sigma_t^2};$$
$$Ek_{it} = k_t = e^{\mu_t+0.5\sigma_t^2}$$

Using the Capital function expressed in (41) the next step is to take expectations

$$\Upsilon k_{it+1} + (1+\pi_{t+1})Xk_{it+1}^{\alpha} = Zk_{it}^{\alpha}$$

$$\Upsilon E(k_{it+1}) + (1+\pi_{t+1})XE[(k_{it+1}^{\alpha})] = ZE[(k_{it})^{\alpha}]$$

This yields

$$\Upsilon k_{t+1} + (1+\pi_{t+1})Xk_{t+1}^{\alpha}e^{0.5\alpha(\alpha-1)\sigma_{t+1}^2} = Zk_t^{\alpha}e^{0.5\alpha(\alpha-1)\sigma_t^2}$$

B Appendix

Appendix B:Variance

This calculation serves to capture the inequality dynamics of the modelled economy. We begin by obtaining the variance of a product of independent random variables namely productivity shock and capital accumulation.

Var(XY)for independent random variables:

$$VarXVarY + VarX(E(Y))^2 + VarY(E(X))^2$$

$$Var(k^{\alpha}_{it+1}\epsilon_{it+1}) = Var(k^{\alpha}_{it+1})Var(\epsilon_{it+1}) + Var(k^{\alpha}_{it+1})(E(\epsilon_{it+1}))^2 + Var(\epsilon_{it+1})E((k^{\alpha}_{it+1}))^2$$

$$= k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}[e^{(\sigma^2_{t+1}\alpha^2)} - 1].(e^{\nu^2} - 1) + k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}[e^{\sigma^2_{t+1}\alpha^2} - 1](1^2) + (e^{nu^2} - 1)[k_{t+1}e^{0.5\sigma^2_{t+1}\alpha(\alpha-1)}]^2$$
$$= k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}[e^{(\sigma^2_{t+1}\alpha^2)} - 1].(e^{\nu^2} - 1) + k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}[e^{\sigma^2_{t+1}\alpha^2} - 1] + (e^{\nu^2} - 1)k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}$$
$$= k^{2\alpha}_{t+1}e^{\sigma^2_{t+1}\alpha(\alpha-1)}[(e^{\nu^2})(e^{\sigma^2_{t+1}\alpha^2} - 1) + (e^{\sigma^2_{t+1}\alpha^2} - 1) + (e^{\nu^2})]$$

Covariance is then also calculated as follows:

$$\begin{aligned}
Cov(k_{it+1}; \epsilon_{it+1}k^{\alpha}_{it+1}) &= E(k_{it+1}\epsilon_{it+1}k^{\alpha}_{it+1}) - E(k_{it+1})E(\epsilon_{it+1}k^{\alpha}_{it+1}) \\
&= E(\epsilon_{it+1})E(k^{1+\alpha}_{it+1}) - E(k_{it+1})E(\epsilon_{it+1})E(k^{\alpha}_{it+1}) \\
\\
&= k^{1+\alpha}_{t+1}[e^{0.5\alpha(\alpha+1)\sigma^2_{t+1}} - e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}] \\
&= k^{1+\alpha}_{t+1}[e^{0.5\alpha(\alpha+1)\sigma^2_{t+1}}e^{-0.5\alpha(\alpha-1)\sigma_{t+1}} - 1]e^{0.5(\alpha-1)\sigma^2_{t+1}} \\
&= k^{1+\alpha}_{t+1}(e^{\alpha\sigma^2_{t+1}} - 1)e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}
\end{aligned}$$

The above calculated are then substituted into the variance equation given in equation

$$Zk^{2\alpha}_t e^{\alpha(\alpha-1)\sigma^2_t}[e^{\alpha^2\sigma^2_t}e^{\nu^2} - 1] = \Upsilon^2 k^2_{t+1}(e^{\sigma^2_{t+1}} - 1) + X^2(1 + \pi_{t+1})^2 k^{2\alpha}_{t+1}e^{\alpha(\alpha-1)\sigma^2_{t+1}}[e^{\alpha^2\sigma^2_{t+1}} - 1]$$

$$+2\Upsilon X k^{1+\alpha}_{t+1}e^{\alpha\sigma^2_{t+1}}e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}$$

which is simplified as;

$$[\Upsilon^2 k^2_{t+1} + X^2(1 + \pi_{t+1})^2 k^{2\alpha}_{t+1}e^{\alpha(\alpha-1)\sigma^2_{t+1}} + 2X\Upsilon(1 + \pi_{t+1})k^{1+\alpha}_{t+1}e^{0.5\alpha(alpha-1)\sigma^2_{t+1}}.e^{\alpha\sigma^2_{t+1}}][\Upsilon^2 k^2_{t+1} +$$

$$X^2(1 + \pi_{t+1})^2 k_{t+1}2\alpha e^{\alpha(\alpha-1)\sigma^2_{t+1}} + 2X\Upsilon(1 + \pi_{t+1})k^{1+\alpha}_{t+1}e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}]^{-1} = e^{\sigma^2_t\alpha^2}e^{\nu^2}$$

Variance is finally given by;

$$\sigma^2_t = \frac{1}{\alpha^2}\ln\left[\frac{\Upsilon^2 k^2_{t+1} + X^2(1 + \pi_{t+1})^2 k^{2\alpha}_{t+1}e^{\alpha(\alpha-1)\sigma^2_{t+1}} + 2X\Upsilon(1 + \pi_{t+1})k^{1+\alpha}_{t+1}e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}e^{\alpha\sigma^2_{t+1}}}{\Upsilon^2 k^2_{t+1} + X^2(1 + \pi_{t+1})^2 k^{2\alpha}_{t+1}e^{\alpha(\alpha-1)\sigma^2_{t+1}} + 2X\Upsilon(1 + \pi_{t+1})k^{1+\alpha}_{t+1}e^{0.5\alpha(\alpha-1)\sigma^2_{t+1}}e^{\nu^2}}\right]$$

C Appendix

Appendix C: Welfare Calculation

$$W_t = \int_0^\infty U(c_{it}; c^o_{it+1}) = \int_0^\infty (\ln c_{it} + \beta \ln c^o_{it+1})di$$
$$W_t = E\ln c_{it} + E\ln c^o_{it+1}$$

Deriving $E\ln c^o_{it+1}$ and $E\ln c_{it}$ from :

$$c_{it} = \frac{\omega}{\beta}\epsilon_{it}k^{\alpha}_{it}$$

$$c^o_{it+1} = \omega\Big[\Upsilon(A\alpha)^{-1}\epsilon^{-1}_{it+1}k^{1-\alpha}_{it+1} + \theta(1+\pi_{t+1})\Big]^{-1}\epsilon_{it}k^{\alpha}_{it}$$

Where $\omega = \dfrac{\beta}{1+\beta}\left(\dfrac{A\theta\alpha\eta + A(1-\alpha)(1-\theta)}{1-\theta}\right)$ and $\Upsilon = (1-\theta)$

For c^o_{it+1} we begin by taking logs

$$\ln(c^o_{it+1}) = \ln\omega - \ln\left[\Upsilon(A\alpha)^{-1}\epsilon^{-1}_{it+1}k^{\alpha}_{it+1} + \theta(1+\pi_{t+1})\right] + \ln\epsilon_{it} + \alpha\ln k_{it}$$

then we take expectations

$$E\ln(c^o_{it+1}) = E\ln\omega - E\ln\left[\Upsilon(A\alpha)^{-1}\epsilon^{-1}_{it+1}k^{\alpha}_{it+1} + \theta(1+\pi_{t+1})\right] + E\ln\epsilon_{it} + \alpha E\ln k_{it} \qquad (C1)$$

We simplify $E\ln\left[\Upsilon(A\alpha)^{-1}\epsilon^{-1}_{it+1}k^{\alpha}_{it+1} + \theta(1+\pi_{t+1})\right]$ to become

$$E\ln[(\frac{s_2}{s_1}\epsilon_{it+1}k^{1-\alpha}_{it+1} + 1)s_1]$$
$$E\ln(\frac{s_2}{s_1}\epsilon_{it+1}k^{1-\alpha}_{it+1} + 1) + E\ln s_1$$
$$E\ln(\frac{s_2}{s_1}\epsilon_{it+1}k^{1-\alpha}_{it+1} + 1) + \ln s_1$$

Following from Getachew and Turnovsky (2020) the formula for $E\ln(x+1)$ is given by

$\ln\left[(x+1)\left(\dfrac{x^2}{(x+1)^2}(e^{\sigma^2}-1)+1\right)^{-0.5}\right]$ which we apply to $E\ln(\frac{s_2}{s_1}\epsilon_{it+1}k^{1-\alpha}_{it+1}+1)$. to get

$$E\ln(\frac{s_2}{s_1}\epsilon_{it+1}k^{1-\alpha}_{it+1} + 1) = \ln\left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}}-1)+1\right)^{-0.5}\right]$$

Substituting into C1 we get the aggregate of $\ln c^o_{it+1}$

$$E\ln(c^o_{it+1}) = \ln\omega - \ln\left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}}-1)+1\right)^{-0.5}\right] - \frac{\nu^2}{2} + 0.5\sigma^2\alpha(\alpha-1) + \ln k^{\alpha}_t \qquad \textbf{C2}$$

For $E\ln c_{it}$ we use $c_{it} = \frac{\omega}{\beta}\epsilon_{it}k^{\alpha}_{it}$ and begin with taking logs;

$$\ln c_{it} = \ln\omega - \ln\beta + \ln\epsilon_{it} + \alpha\ln k_{it}$$

then expectation;

$$E\ln c_{it} = E\ln\omega - E\ln\beta + E\ln\epsilon_{it} + \alpha E\ln k_{it}$$
$$E\ln c_{it} = \ln\omega - \ln\beta - \frac{\nu^2}{2} + \ln k_t^{\alpha} + 0.5\sigma_{t,k}^2\alpha(\alpha-1) \qquad C3$$

Combining C2 and C3 gives us the welfare.

$$W_t = 2\ln\omega - \ln\beta - \nu^2 + 2\ln k_t^{\alpha} + \sigma_{t,k}^2\alpha(\alpha-1) - \ln\left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}}-1)+1\right)^{-0.5}\right]$$

The steady state welfare is given below

$$W^s = 2\ln\omega - \ln\beta - \nu^2 + 2\ln k^{\alpha} + \sigma^2\alpha(\alpha-1) - \ln\left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma^2}-1)+1\right)^{-0.5}\right]$$

Aggregating welfare over all generation at a discount rate of R

$$W^s = \sum_{t=0}^{\infty} W^s(1+R)^{-t}$$

$$W \approx \frac{1+R}{R}\left(2\ln\omega - \ln\beta - \nu^2 + 2\ln k^{\alpha} + \sigma^2\alpha(\alpha-1) - \ln\left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma^2}-1)+1\right)^{-0.5}\right)$$

D Appendix

Appendix D: Derivation of Total Consumption

In section we obtain c_t and c_{t+1}^o using $\ln c_t = \ln E(c_{it}) - 0.5\sigma_c^2$ and $lnc_{t+1}^o = \ln E(c_{it+1}^o) - 0.5\sigma_{c_{t+1}}^2$. In Appendix C $\ln E(c_{it})$ and $\ln E(c_{it+1}^o)$ are already established and therefore we begin with solving for the variance of both young and old's consumption following the formula utilised by (Getachew & Turnovsky, 2020) to obtain the variance of $\ln(x+1)$.

$$c_{it} = \frac{\omega}{\beta}\epsilon_{it}k_{it}^{\alpha} \qquad D1$$

$$c_{it+1}^o = \omega\left[\Upsilon(A\alpha)^{-1}\epsilon_{it+1}^{-1}k_{it+1}^{1-\alpha} + \theta(1+\pi_{t+1})\right]^{-1}\epsilon_{it}k_{it}^{\alpha} \qquad D2$$

Begin by taking logs of of the above equations for c_{it} and c_{it+1}^o and then subsequently taking variance as shown below: Solving using D1:

$$\ln c_{it} = \ln\omega - \ln\beta + \ln\epsilon_{it} + \alpha\ln k_{it}$$

$$Var(\ln c_{it}) = Var(\ln \omega) - Var(\ln \beta) + Var(\ln \epsilon_{it}) + Var(\alpha \ln k_{it})$$

$$\sigma_{c,t} = \nu^2 + \alpha^2 \sigma^2_{k,t} \qquad D3$$

And for D2:

$$\ln c^o_{it+1} = \ln \omega - \ln \left[\Upsilon (A\alpha)^{-1} \epsilon^{-1}_{it+1} k^{1-\alpha}_{it+1} + \theta(1 + \pi_{t+1}) \right] + \ln \epsilon_{it} + \ln k^{\alpha}_{it}$$

$$Var(\ln c^o_{it+1}) = Var(\ln \omega) - Var(\ln \left[\Upsilon (A\alpha)^{-1} \epsilon^{-1}_{it+1} k^{1-\alpha}_{it+1} + \theta(1 + \pi_{t+1}) \right]) + Var(\ln \epsilon_{it}) + Var(\alpha \ln k_{it})$$

Following from Appendix C, we simplify $\ln \left[\Upsilon (A\alpha)^{-1} \epsilon^{-1}_{it+1} k^{\alpha}_{it+1} + \theta(1 + \pi_{t+1}) \right]$ to become

$$\ln(\frac{s_2}{s_1} \epsilon^{-1}_{it+1} k^{1-\alpha}_{it+1} + 1) + \ln s_1 \qquad D5$$

Where $s_2 = \Upsilon (A\alpha)^{-1}$ and $s_1 = \theta(1 + \pi_{t+1})$.

Using Appendix A in (Getachew & Turnovsky, 2020), variance of D3 is given by :

$$\sigma^2 = \ln \left(u^2 (e^{\sigma^2_1} - 1)/(u + 1)^2 + 1 \right)$$

Where $u = \frac{s_2}{s_1} k^{1-\alpha}_{t+1} e^{0.5\alpha(\alpha-1)\sigma_{t+1,k}}$

Therefore variance of D5 becomes:

$$\ln \left(u^2 (e^{\sigma^2_{k,t}} - 1)/(u + 1)^2 + 1 \right)$$

and D3 simplifies to

$$\sigma^2_{c^0,t+1} = \nu^2 + \alpha^2 \sigma^2_{k,t} - \ln \left(u^2 (e^{\sigma^2_{k,t}} - 1)/(u + 1)^2 + 1 \right) \qquad D6$$

As the intention is to solve for c_t and c^o_{t+1}, we substitute into the $\ln c_t = \ln E(c_{it}) - 0.5\sigma^2_c$ and $lnc^o_{t+1} = \ln E(c^o_{it+1}) - 0.5\sigma^2_{c_{t+1}}$ using C2, C3 D6 and D3. Exponentiating both equations gives c_{it} and c^o_{it+1} as shown below.

$$c_t = \frac{\omega}{\beta} k^{\alpha}_t e^{-(0.5\alpha^2 \sigma^2_{k,t} + \nu^2)}$$

$$c^o_{t+1} = \omega k^{\alpha}_t s^{-1}_1 e^{0.5\alpha(\alpha-1)\sigma^2_{k,t} - \frac{\nu^2}{2}} \left[\frac{s_2}{s_1}(u + 1) \left(\frac{u^2}{(u + 1)^2} (e^{\sigma_{t,k}} - 1) + 1 \right)^{0.5} \right] \left(\frac{u^2}{(u + 1)^2} (e^{\sigma_{t,k}} - 1) + 1 \right)^{-0.5} e^{-0.}$$

$$c^o_{t+1} = \omega k^{\alpha}_t s^{-1}_1 e^{-(0.5\alpha\sigma^2_{k,t} + \nu^2)} \left[\frac{s_2}{s_1}(u + 1) \left(\frac{u^2}{(u + 1)^2} (e^{\sigma_{t,k}} - 1) + 1 \right) \right]$$

Having established the equations for aggregate young age and old age consumption we go on to get total consumption by adding the two.

$$\mathbf{C} = c_t + c^o_{t+1}$$

$$\mathbf{C} = \frac{\omega}{\beta} k_t^{\alpha} e^{-(0.5\alpha^2\sigma^2_{k,t}+\nu^2)} + \omega k_t^{\alpha} s_1^{-1} e^{-(0.5\alpha\sigma^2_{k,t}+\nu^2)} \left[\frac{s_2}{s_1}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}}-1)+1\right)\right]$$

$$\mathbf{C} = \omega k_t^{\alpha} e^{-(0.5\alpha\sigma^2_{k,t}+\nu^2)} \left[\frac{1}{\beta} + \frac{s_2}{s_1^2}(u+1)\left(\frac{u^2}{(u+1)^2}(e^{\sigma_{t,k}}-1)+1\right)\right]$$

E Appendix

E: Calibration of money growth rate (η)

Money supply growth is chosen by the Central Bank based on the set inflation target as in Heer and Maußner (2011). In South Africa there is a set range between 3-6% and therefore for the the purpsoes of this study we apply the lower limit of 3% in the baseline calibration calculation.

$$(1+\pi^T)^4 - 1$$

where π^T is the targeted inflation.

$$(1+0.03)^4 - 1 = 0.12550881$$